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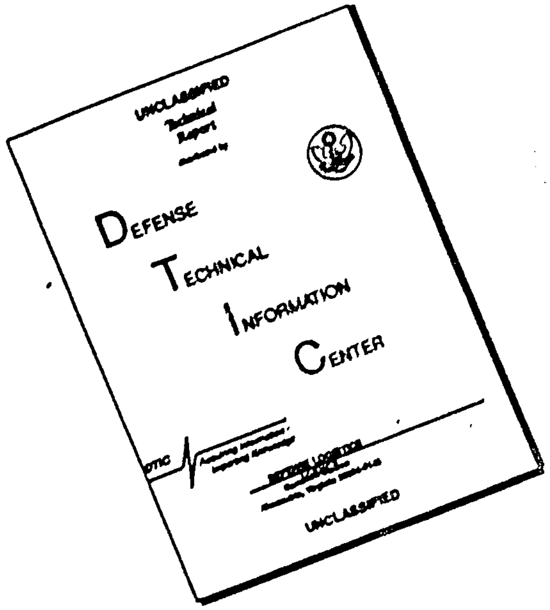

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PRINCETON UNIVERSITY
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COMBUSTION INSTABILITY

IN

LIQUID PROPELLANT ROCKET MOTORS

TECHNICAL REPORT:

TRANSVERSE WAVE AND ENTROPY WAVE COMBUSTION INSTABILITY IN LIQUID PROPELLANT ROCKETS

Aeronautical Engineering Report No. 380

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SUMMARY

The equations of motion governing the unsteady flow in a liquid bipropellant rocket are derived. These are utilized in an analytical investigation of two mechanisms which are capable of producing linear combustion instability in the high and intermediate frequency ranges, typified by the appearance of transverse waves and entropy waves respectively.

The characteristic equation of each rocket system is derived, and it is shown how the stability limits may be determined for a particular rocket motor.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

Superscripts

*	indicates that the quantity is dimensional
(0), (1)	indicate the solutions obtained from the zeroth or first iteration
'	denotes a small perturbation
$\overline{}$	over a quantity denotes steady state

Subscripts

b	indicates a burned gas quantity
c	denotes a chamber dimension
c.v.	denotes a cavitating venturi injector
e	indicates that the quantity is evaluated at the exhaust end of the chamber
f	denotes the fuel line
i	indicates an injected quantity
Im	denotes the imaginary part of a quantity
ℓ	indicates a quantity belonging to the liquid phase
M.I.	denotes a matched impedance injector
o	indicates that the quantity is evaluated at the injector end
ox	denotes the oxidizer line
r	denotes the radial component
R	denotes a quantity evaluated at the propellant reservoir
Re	denotes the real part of a quantity
s	indicates the isentropic stagnation value of the quantity
sub	denotes the subsonic part of the deLaval nozzle
z	denotes the axial component
θ	denotes the tangential component

$O(\ )$	order of magnitude of the quantity in parentheses
a	dimensionless characteristic time of the propellant feedline
a_1, a_2	functions defined in Eq. 8.5
A^*	dimensional area
$A(\omega)$	function defined in Eq. C 16
$A(z)$	function defined in Eq. C 7
$A_1(\omega), A_2(\omega), A_3(\omega), \cdots A_8(\omega)$	functions defined in Eq. C 18
$\mathcal{A}, \mathcal{B}, \mathcal{C}$	nozzle admittance coefficients defined in Eq. 7.7
b_1, b_2, b_3	functions defined in Eq. 13.5
b_4, b_5	functions defined in Eq. 13.14
$B(z), D(z), E(z), F(z), G(z)$	functions defined in Eq. 5.41 for transverse wave analysis and in Eq. 10.17 for entropy wave analysis
$B(\omega)$	function defined in Eq. C 5
$B_1(\omega), B_2(\omega), B_3(\omega), \cdots B_8(\omega)$	functions defined in Eq. C 10
c_0^*	dimensional speed of sound in stagnant burned gas at the injector end
$C(\omega)$	function defined in Eq. C 15
$D(\omega)$	function defined in Eq. C 4
e^*	dimensional internal energy per unit mass
E_a	dimensionless activation energy
E_c, E_s, G_c, G_s	functions defined in Eq. 8.1
f	overall rate of the conditioning processes
$g(\omega), G(\omega), H(\omega), J(\omega), M(\omega)$	injector transfer functions
h^*	dimensional enthalpy per unit mass
$\mathcal{h}(z)$	function defined in Eq. 5.45 or Eq. 10.21
h_1, h_2, h_3	complex functions defined in Eq. 13.1
I_{sp}	specific impulse

k^*	dimensional drag coefficient of the liquid droplets
$\mathcal{K}$	non-dimensional enthalpy gradient
K	non-dimensional velocity gradient
ℓ^*	dimensional length
L^*	dimensional length of the combustion chamber
$L(\omega)$	function defined in Eq. C 3
$\dot{m}^*$	dimensional mass flow rate
$\delta\dot{m}$	dimensionless fractional mass flow rate
$\mathfrak{m}$	mixture ratio interaction index defined in Eq. 11.12
M	Mach number
$M(z)$	function defined in Eq. 5.37
$\mathfrak{n}$	pressure interaction index
$\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3$	indices defined in Eqs. 11.3 and 11.7
p^*	dimensional pressure
r^*	dimensional radius
r	mixture ratio, ratio of mass of oxidizer to mass of fuel
$\mathcal{R}^*$	dimensional gas constant
R_c, R_s	functions defined in Eq. 8.1
$s^* = \Lambda^* + i\Omega^*$	a dimensional complex quantity which is the root of the characteristic equation for oscillations with exponential time dependence
S^*	dimensional entropy
s_{nh}	h^{th} zero of the derivative of the Bessel function of order n
t^*	dimensional time
t'	dimensionless dummy variable in time
T^*	dimensional temperature
U_c, U_s, V_c, V_s	functions defined in Eq. 8.1
$U(z), V(z), W(z), X(z), Y(z), Z(z)$	functions defined in Eqs. 5.32 and 5.35 or Eqs 10.11 and 10.13

V^*	dimensional velocity
$\bar{w}$	dimensionless steady state mass flux
y	physical factor
z^*	dimensional axial distance measured from the injector end of the combustion chamber
z'	dimensionless dummy variable in space
$\mathfrak{z}_A, \mathfrak{z}_B$	symbols defined in Eq. C 6 and C 17
α_n, β_n	nozzle admittance coefficients defined in Eq. 12.14
β	non-dimensional frequency, (= ω_{fig})
γ	ratio of specific heats of the burned gas
ρ^*	dimensional density
$\bar{\tau}_i$	dimensionless insensitive time lag
τ	dimensionless sensitive time lag
τ_t	dimensionless total time lag
$\bar{\delta}$	dimensionless critical value of the sensitive time lag
$\bar{\delta}_1, \bar{\delta}_2$	sensitive time lags defined in Eqs. 13.11 and 13.12
$\bar{\xi}, K, \chi$	dimensionless coordinates of insensitive space lag
Λ	dimensionless amplification coefficient
Ω	dimensionless angular frequency
ω	dimensionless critical value (neutral) angular frequency
ϕ	instantaneous mass rate of gas production
θ	tangential coordinate
Ψ	radial functional dependence
Φ	tangential functional dependence
ν	axial dependence of gas velocity perturbation
η	axial dependence of liquid velocity perturbation
δ	axial dependence of gas density perturbation
ξ	axial dependence of liquid density perturbation
φ	axial dependence of pressure perturbation

$\frac{dq}{d\bar{z}}$	axial dependence of source distribution
ϵ	axial dependence of entropy distribution
Π, Ξ	functions defined in Eq. C 3

I. INTRODUCTION TO THE THEORY

1. Introduction

Coupled with the demand for liquid propellant rocket powerplants of ever higher performance and consequently the development of high energy release combustion chambers, there has arisen a problem which has continued to grow in importance. The problem, which can arise in any ducted burner, is the phenomenon of combustion instability, and although a precise universally accepted definition does not exist, it is generally considered to consist of regular periodic oscillations in combustion chamber pressure which are maintained in some manner by the combustion process. These pressure variations are observed experimentally to cover a frequency range from 10 to 12,000 cycles per second, at amplitudes which vary from ten to one hundred percent of chamber pressure.

Oscillations of pressure, and hence of all dependent thermodynamic and fluid dynamic variables, are highly undesirable in general, since mechanical or thermal failure, and control malfunction may ensue shortly. The varying chamber pressure, as well as the varying thrust will unduly stress or fatigue both the chamber and its mounts causing mechanical failure. The heat transfer to the wall can be increased several hundred precent, due to the presence of high frequency oscillations, which may be sufficient to cause rapid deterioration and burnout of the chamber wall. Finally, even if the rocket motor can withstand the vibration and heat transfer, severe secondary oscillations may be set up in the delicate guidance and control system which will destroy its effectiveness. Any, or all, of these effects will result in failure of the propulsion unit and can eliminate it from consideration for inclusion in a system which requires reliability, accuracy and dependability. Since the presence of combustion instability affects the life and reliability of the rocket motor, it is necessary that we gain an understanding of the

fluid dynamic processes so that the conditions which promote this detrimental form of combustion may be determined and consequently avoided, controlled or eliminated.

Although there have been other attempts at classification, (for example, see Refs. 5 and 11) the frequency spectrum in which combustion instability occurs, can be conveniently separated into three parts, denoted low, intermediate and high frequency instability, respectively. This appears to be a natural division since the coupled wave processes, governing each of these three types, are different and hence lead to different characteristic frequencies.

If all the gas in the chamber surges periodically, large inertias are involved and low frequency oscillations result. These low frequency instabilities, commonly referred to as "chugging", have frequencies ranging from 10 to 200 cycles per second, and have been demonstrated both theoretically (Refs. 2 and 3) and experimentally (Ref. 10) to depend primarily upon the coupling with the propellant feedlines. Since this case has already been thoroughly explored, and since the instability may be eliminated in most instances by increasing the injector impedance, that is, by increasing the injector pressure drop, we need not consider this type further.

Intermediate frequency instabilities have been observed experimentally (Refs. 9 and 16) and generally occur at frequencies of several hundred cycles per second. It has been postulated (see Section 2) that the characteristic frequency may be attributed to the presence of entropy waves in the chamber. This type has not yet received a comprehensive analytical treatment, until now.

High frequency or "screaming" oscillations are generally associated with various acoustic modes of the chamber and occur at frequencies between several hundred and several thousand cycles per second depending on the mode, chamber geometry, and exhaust nozzle. That is, experimental observation

indicates that the intense shrill sound that occasionally issues from a rocket is produced by high frequency pressure oscillations characterized by frequencies in the neighborhood of the organ-pipe resonances of the chamber. These instabilities were initially found to be of the longitudinal type, but with more adequate instrumentation, it was discovered that transverse modes were also present. The most frequently encountered transverse modes are forms of the first tangential, or "spinning-sloshing" mode, in which the pressure waves propagate diametrically or tangentially across the chamber. Transverse modes of instability in rocket chambers have not as yet received comprehensive analytical treatment, until now.

In all of these cases, if oscillations are to be maintained, there must be some process of coordination which periodically feeds sufficient energy into the oscillating gas system to sustain the process. Of course, in order to have a closed cycle, the gas dynamic system must induce the combustion process to release energy at the proper time phase-wise during each cycle; thus, any theoretical model must demonstrate a closed loop.

The analytical investigation presented here consists of two parts. The first deals with the treatment of high frequency transverse mode instability, in which the first tangential mode is of primary interest. The second deals with a theory of combustion instability in the intermediate frequency range which is characterized by the appearance of entropy waves in the chamber.

Although analytical treatment of the problem of combustion instability in liquid propellant rockets is comparatively recent, an extensive literature on the subject already exists. Undoubtedly, the most comprehensive treatment of the subject, to date, is due to Crocco and Cheng. Because of the thoroughness of the analysis and discussion appearing in their recent treatise, (Ref. 18), the reader is advised that the latter constitutes the primary reference for this thesis.

2. Status of the Theory

In their recent survey, (Ref. 13), Putnam and Dennis point out that of all types of combustion oscillations which have been observed, the organ-pipe type, in which the wavelength of the oscillation is related to the dimensions of the chamber, has the oldest history. It seems that Higgins produced an acoustic oscillation or "singing flame" in 1777 by surrounding a diffusion flame with a large duct open at both ends.

Since that time, thermo-acoustic oscillations have been observed in many different pieces of laboratory and industrial equipment, for example, detonation tubes and gas-fired units, and consequently, many investigators have attempted to analyze the mechanisms involved.

One of the earliest analyses, due to Rayleigh (Ref. 1), was one in which he advanced a criterion without formal proof, that for the excitation of thermally driven oscillations, there must be a fluctuating heat release within the medium, such that it has a component in phase with the varying component of the pressure at the position of heat release. Note that this implies that heat release at a pressure node cannot contribute to thermo-acoustic oscillations. In a crude sense, Rayleigh's criterion may be considered to be nothing more than the restatement of the conditions for a closed thermodynamic cycle. That is, if heat is supplied to a medium at high pressure and rejected at low pressure, then piston work may be obtained. Presumably, this net work is utilized in driving the pressure waves.

As previously stated, many contributors have analysed the fluid dynamics of various combustion driven oscillations, for example, see Refs. 12, 13, 14, 15 and 20. However, since these and other published works do not bear directly on the specific problem which concerns us, namely, combustion instability in liquid propellant rockets, we will now proceed to review some of the concepts leading to the theory presented in the following sections. We

will begin with the evolution of the time lag concept in rocket motors.

First, we recognize that the normal process of combustion in a rocket motor is of a highly turbulent nature, and hence there are time- and space-wise fluctuations of the pressure and dependent thermodynamic and fluid-dynamic variables throughout the chamber. If the fluctuations have small amplitudes, then the combustion is considered smooth, and when they have large amplitudes, it is called rough burning. However, these terms are qualitative at best, inasmuch as the amplitude of the oscillations alone is an insufficient criterion for classifying the stability of the combustion.

Random fluctuations, i.e. those which do not have a characteristic frequency, exhibit the attributes of turbulence and in certain cases will not be detrimental to the practical operation of the rocket. That is, the time averaged exciting forces are negligible and the combustion is rough but stable. When, however, periodic fluctuations are present such that one or more components of the frequency are predominant and grow in amplitude, the integrated effects will be non-zero and mechanical or thermal failure may follow in short order. Accordingly, we note that the occurence of rough and detrimental combustion, given the name unstable combustion, is characterised by oscillations with well-defined frequencies whose amplitude is limited only by the damping of the system. This distinction between rough but stable combustion, and unstable combustion was advanced by Crocco several years ago.

In order that unstable combustion exist as defined above, some coordinating influence, which is capable of amplifying a random disturbance, must be present. In this connection, in 1941, von Karman's group advanced the concept of a combustion time lag, or delayed instantaneous combustion. This was defined as the time between the injection of a propellant element and its evolution into combustion products, which was assumed to occur instantaneously, after a certain delay.

Gunder and Friant, (Ref. 2) were the first to present a formalized analysis applying this concept to a treatment of low frequency combustion instability. They postulated a constant value of the time lag and were able to demonstrate that under certain circumstances, this time lag could provide the necessary coupling between the rocket chamber and the propellant feed system which would result in unstable amplification of a random pressure disturbance. Other, more elaborate, low frequency analyses followed, including the early work of Summerfield (Ref. 3) who established the theoretical limits of low frequency stability in his analysis which included the effects of chamber capacitance and feeding system inertia. However, since each of these retained the constant time lag concept, the variation of the rate of burned gas generation, or source of driving energy, could be modified only by the rate of supply of propellant to the chamber, which depended entirely upon the sensitivity of the feeding system to chamber pressure oscillations.

Now although Crocco was willing to retain the concept of an instantaneous transformation of liquid propellant into gaseous products, he reasoned that the specific rate of conversion of unburned propellant elements into combustion products depends on the sensitivity of the activation processes to chamber oscillations, and in 1951 he introduced the concept of the sensitive time lag. Thus the total time lag τ_t , during which various physico-chemical processes occur, was considered to consist of two parts, a constant part $\bar{\tau}_i$ during which mechanical processes insensitive to the thermodynamic states of the surrounding gas take place, and a variable sensitive part τ during which activation processes occur which are sensitive to the oscillations occurring in the surrounding gas. Simply written:

$$\tau_t = \bar{\tau}_i + \tau \qquad 2.1$$

where the bar denotes steady state. And then to relate the sensitive time lag to the rates of the conditioning processes, Crocco wrote an integral equation for an element burning at time t,

$$\int_{t-\tau}^{t} f(t')\,dt' = \text{const.} \tag{2.2}$$

where f is the overall rate of the conditioning processes, and the integral must be evaluated following each individual propellant element. The rate f varies along the path of integration, and upon correlating all of the physical factors which cause this variation, to the pressure, Crocco obtained the relation

$$f = \bar{f}\left(1 + n\frac{p'}{\bar{p}}\right) \tag{2.3}$$

which is valid for small amplitudes of oscillation. The quantity n which is actually a well-defined mathematical quantity (see Ref. 18) was called an interaction index, and was assumed to be a characteristic of a given propellant combination. Crocco observed that actually, of course, sensitive and insensitive time lags are physically inseparable and occur simultaneously. Hence Eq. 2.1 should be considered as a schematic representation for the actual events. On combining Eqs. 2.2 and 2.3 we observe that the sensitive time lag exhibits inverse pressure dependence, since there is readily obtained:

$$\bar{\tau}\bar{p}^{\,n} = \text{const.} \tag{2.4}$$

Application of the sensitive time lag concept to the treatment of combustion instability was first made by Crocco (Ref. 4) for the cases of low frequency instability in monopropellant and bipropellant motors, and roughly, for longitudinal high frequency instability. The term intrinsic instability was introduced to indicate that due to the coupling between the pressure oscillations and combustion processes, through the medium of the sensitive time lag, instability can occur even if the injection system delivers a constant flow.

A series of papers by Crocco and Cheng dealing with a more refined treatment of the low and high frequency longitudinal modes of combustion instability followed and was recently assembled in their monograph (Ref. 18). In this same reference, it was pointed out that theoretical analyses were lacking for two important mechanisms. Accordingly, we will consider the treatment of these two distinct problems, transverse wave and entropy wave combustion instability.

In both cases to be analyzed here, the treatment is restricted to linear instability, where the perturbations are sufficiently small so that second order terms may be considered negligible. It is recognized that fully-developed combustion instability is generally a non-linear phenomenon, characterised by the presence of shock waves, but as has been observed experimentally in a great many cases, instability will oftimes result from the progressive amplification of small disturbances. Thus, when we determine stability limits, it means that within those limits, the system will be stable to small disturbances. If a given system is linearly stable, and no large disturbances are applied, then self-amplification cannot drive the system to instability, and no high-amplitude pressure oscillations can appear.

Let us now describe the physical system and the mechanisms which are involved. At the high frequencies normally encountered in transverse mode instability, the injection system cannot respond to chamber pressure oscillations. Thus, for instability to exist, it must be of the intrinsic variety, and a coordinating process must be present, so that oscillations of the rate affecting factors will produce organized oscillations of the burning rates, which will provide the necessary exciting force to maintain the coordinating process.

In our investigation of transverse modes, we will retain Crocco's model of the sensitive time lag as a suitable mechanism, and we will

investigate the stability of three-dimensional perturbations with exponential time dependence, in a general rocket system consisting of a rigid cylindrical chamber with a liquid propellant injector and a fixed converging-diverging exhaust nozzle. It should be observed that due to the presence of the exhaust nozzle, purely transverse oscillations cannot be present. If a transverse perturbation appears, a longitudinal wave will be reflected in the subsonic portion of the nozzle, and thus a form of combined mode must exist, in contradistinction to what occurs in a cylindrical chamber terminated by plane closed ends.

To explain the experimentally observed occurrence of combustion instability in the intermediate frequency range (Ref. 9), another mechanism must be considered. In this case, the frequencies are too high to be the result of chamber and feed system coupling, and they are too low to correspond to a mode of resonance of the gases in the chamber. The proposed mechanism depends on the presence of entropy waves in the chamber, and in some cases involves a special form of coupling between the injection system and the chamber.

Entropy waves may be produced by chamber pressure oscillations in one of two ways, each of which is capable of reinforcing the other. However, in a bipropellant rocket, the primary cause arises when one obtains a variation of the mixture ratio about its mean value. This occurs when the oxidizer and fuel injectors respond differently to chamber pressure oscillations, thus producing a stream of propellant elements with an off-mixture ratio. If the mixture ratio is oscillating, the temperature and the entropy of the products of combustion at a given station will also oscillate. For typical rockets designed for maximum thrust, both become [illegible] or smaller as the oxidizer-fuel ratio increases or decreases. (see Fig. 2.1)

As a result, if one examines the conditions in the chamber of this rocket at any given instant of time, one observes that the gas entropy

distribution has a wave-like pattern which moves downstream with the gas velocity. The entropy oscillation reflects pressure waves at the exhaust nozzle which travel upstream with the speed of sound to the injector face. This closes the feedback loop allowing the process to start all over again. The explanation of this mechanism was first given roughly by Berman and Cheney (Ref. 9), and has since been enlarged upon (Refs. 16, 18). In the mathematical treatment in which the stability limits for this mechanism will be determined, Crocco's model of the sensitive time lag is extended to allow for the functional dependence of the rate of the conditioning processes on the mixture ratio. The treatment of entropy wave instability appears in Sections 9 through 13.

II. TRANSVERSE WAVE INSTABILITY

3. Formulation of the Governing Equations

In a generalized treatment of combustion instability, we are concerned with the three-dimensional motion of a gas, containing a distribution of droplets of liquid propellant, which flows through a rocket chamber. The droplets which burn at different locations in the chamber will then correspond to a distribution of sources of mass, momentum and energy. Let us formulate the conservation laws for this two-phase flow.

We let ρ denote the density of the gas, defined as the mass of gas per unit volume of gas, and let ρ_ℓ denote the density of the liquid droplets, defined as the mass of liquid per unit volume of gas, that is, we will neglect the volume of the droplets with respect to the gaseous volume. Then letting a * denote a dimensional quantity, the conservation of mass becomes

$$\frac{\partial}{\partial t^*}(\rho^* + \rho_\ell^*) + \nabla^* \cdot (\rho^* \underline{V}^* + \rho_\ell^* \underline{V}_\ell^*) = 0 \qquad 3.1$$

where $\underline{V}^*$ and $\underline{V}_\ell^*$ are the gas velocity and liquid droplet velocity respectively, t^* is the time and $\nabla \cdot^*$ is the divergence operator.

Now note that the continuity equation may be treated in another way. If ϕ^* denotes the instantaneous rate per unit volume at which gas is generated at any location in the chamber, so that ϕ^* corresponds to a source distribution for the gas phase and simultaneously a sink for the liquid phase, then continuity may also be written:

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \underline{V}^*) = \phi^* = -\frac{\partial \rho_\ell^*}{\partial t^*} - \nabla^* \cdot (\rho_\ell^* \underline{V}_\ell^*) \qquad 3.2$$

If viscosity is neglected, the conservation of momentum in this two-phase system takes the form:

$$\frac{\partial}{\partial t^*}\left(\rho^*\underline{V}^* + \rho_\ell^*\underline{V}_\ell^*\right) + \rho^*\underline{V}^*(\nabla^*\cdot\underline{V}^*) + (\underline{V}^*\cdot\nabla^*)\rho^*\underline{V}^*$$
$$+\rho_\ell^*\underline{V}_\ell^*(\nabla^*\cdot\underline{V}_\ell^*) + (\underline{V}_\ell^*\cdot\nabla^*)\rho_\ell^*\underline{V}_\ell^* = -\nabla^* p^* \qquad 3.3$$

which is somewhat more complicated than the familiar Euler equation.

The energy changes, must obey the first law of thermodynamics, i.e., the work done on the gas, plus the heat added to the gas must equal the change of energy of the gas. The several forms of energy which will be considered here include kinetic, internal and chemical energy, however, the work done by viscous stresses and heat transferred by conduction or diffusion will be neglected. By definition,

$$e_s^* = e^* + \frac{V^{*2}}{2}$$
$$h_s^* = h^* + \frac{V^{*2}}{2} \qquad 3.4$$

and noting that

$$h_\ell^* \cong e_\ell^* \qquad 3.5$$

since for the liquid phase the internal energy and the enthalpy are very nearly the same, and the corresponding common value h_ℓ^* is intended to include the chemical energy of the propellants, we obtain the following equivalent forms of the energy equation:

$$\frac{\partial}{\partial t^*}\left(\rho^* e_s^* + \rho_\ell^* h_{\ell s}^*\right) + \nabla^*\cdot\left(\rho^* h_s^*\underline{V}^* + \rho_\ell^* h_{\ell s}^*\underline{V}_\ell^*\right) = 0 \qquad 3.6$$

and

$$\frac{\partial}{\partial t^*}\left(\rho^* h_s^* + \rho_\ell^* h_{\ell s}^*\right) + \nabla^*\cdot\left(\rho^* h_s^*\underline{V}^* + \rho_\ell^* h_{\ell s}^*\underline{V}_\ell^*\right) = \frac{\partial p^*}{\partial t^*} \qquad 3.7$$

Noting that h^* is a function of the temperature, the four equations 3.1, 3.2, 3.3, and 3.7 contain eight unknowns $\rho^*, \underset{\rightarrow}{V}^*, p^*, T^*, \phi^*, \rho_\ell^*, \underset{\rightarrow}{V_\ell}^*$ and h_ℓ^*, hence four additional equations are required. First we have the equation of state for the gas phase:

$$p^* = \rho^* \mathcal{R}^* T^* \qquad 3.8$$

A second equation relating the burning rate ϕ^* to the other quantities will be derived later.

A third equation is obtained from the dynamic behavior of the droplets:

$$\frac{d\underset{\rightarrow}{V_\ell}^*}{dt^*} = \frac{\partial \underset{\rightarrow}{V_\ell}^*}{\partial t^*} + (\underset{\rightarrow}{V_\ell}^* \cdot \nabla^*)\underset{\rightarrow}{V_\ell}^* = k(\underset{\rightarrow}{V}^* - \underset{\rightarrow}{V_\ell}^*) \qquad 3.9$$

which assumes that the force exerted by the gases on the liquid droplets is inversely proportional to the Reynolds Number.

The fourth equation could be obtained from the heat balance of the droplets which would yield an expression relating h_ℓ^* and h^*. We shall, however, neglect the heat transfer between the gases and the liquid droplets by taking (see Ref. 18)

$$\frac{dh_{\ell s}^*}{dt^*} = \frac{\partial h_{\ell s}^*}{\partial t^*} + (\underset{\rightarrow}{V_\ell}^* \cdot \nabla^*) h_{\ell s}^* = 0 \qquad 3.10$$

This implies that when we follow the motion of a particular droplet, the value of $h_{\ell s}^*$ is conserved. In other words, the droplet retains the value of $h_{\ell s}^*$ with which it was injected. We may then write:

$$h_{\ell s}^* = h_\ell^* + \frac{V_\ell^{*2}}{2} = h_{\ell s_o}^* = h_{\ell_o}^* + \frac{V_{\ell o}^{*2}}{2} \qquad 3.11$$

where $h_{\ell_o}^*$ and $\frac{V_{\ell o}^{*2}}{2}$ are the values of the propellant enthalpy and kinetic energy at the injector head, and we remark that these quantities may be related to the pressure at the injector by means of the injector response equations. If the injection system does not respond to chamber pressure

oscillations, the mixture ratio remains constant, therefore $h_{\ell_0}^*$ must likewise be constant, and unless the injection velocity is modulated, $\frac{V_{\ell_0}^{*2}}{2}$ is also constant. In general, the mixture ratio and injection velocity are not constant, and each droplet retains a different characteristic stagnation enthalpy right through the moment of its conversion into burned gas.

We may now proceed to non-dimensionalize the foregoing equations by using as reference values $\bar{p}_0^*$, $\bar{\rho}_0^*$, $\bar{T}_0^*$ and $\bar{c}_0^* = \sqrt{\gamma \mathcal{R}^* T_0^*}$, which are the steady state values of pressure, density, temperature of the gas and the velocity of sound in the gas at the injector face. Note that since $\bar{\underset{\rightarrow}{V}}^*$ must be zero at the injector face, the reference values correspond to stagnation values of the respective quanitities. In the study of transverse waves, it is convenient to use the chamber radius r_c^* as the reference length, whereas in the study of longitudinal waves, the combustion chamber length is a more suitable reference length. We will begin by selecting r_c^* as the measure of length.

Let

$$\nabla = r_c^* \nabla^*, \quad t = \frac{\bar{c}_0^* t^*}{r_c^*}, \quad p = \frac{p^*}{\bar{p}_0^*}, \quad \rho = \frac{\rho^*}{\bar{\rho}_0^*}$$

$$T = \frac{T^*}{\bar{T}_0^*}, \quad \rho_\ell = \frac{\rho_\ell^*}{\bar{\rho}_0^*}, \quad \underset{\rightarrow}{V} = \frac{\underset{\rightarrow}{V}^*}{\bar{c}_0^*}, \quad \underset{\rightarrow}{V_\ell} = \frac{\underset{\rightarrow}{V_\ell}^*}{\bar{c}_0^*} \qquad 3.12$$

$$h = \frac{\gamma - 1}{\gamma \mathcal{R}^* T_0^*} h^*, \quad \phi = \frac{r_c^* \phi^*}{\bar{\rho}_0^* \bar{c}_0^*}, \quad k = \frac{k^* r_c^*}{\bar{c}_0^*}$$

where γ is the ratio of specific heats which is assumed to be constant over the range of variation of T^*. This assumption yields $dh = dT$ in terms of the non-dimensional variables. The non-dimensional equations may now be written:

$$\frac{\partial}{\partial t}(\rho + \rho_\ell) + \nabla \cdot (\rho \underset{\rightarrow}{V} + \rho_\ell \underset{\rightarrow}{V_\ell}) = 0 \qquad 3.1a$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{V}) = \phi = -\frac{\partial \rho_\ell}{\partial t} - \nabla\cdot(\rho_\ell \underline{V_\ell}) \tag{3.2a}$$

$$\frac{\partial}{\partial t}(\rho \underline{V} + \rho_\ell \underline{V_\ell}) + \rho \underline{V}(\nabla\cdot\underline{V}) + (\underline{V}\cdot\nabla)\rho\underline{V} + \rho_\ell \underline{V_\ell}(\nabla\cdot\underline{V_\ell}) + (\underline{V_\ell}\cdot\nabla)\rho_\ell \underline{V_\ell} = -\frac{1}{\gamma}\nabla p \tag{3.3a}$$

$$\rho\left[\frac{\partial h_s}{\partial t} + (\underline{V}\cdot\nabla)h_s\right] = \frac{\gamma-1}{\gamma}\frac{\partial p}{\partial t} - \phi(h_s - h_{\ell s}) \tag{3.7a}$$

$$p = \rho T \tag{3.8a}$$

$$\frac{d\underline{V_\ell}}{dt} = \frac{\partial \underline{V_\ell}}{\partial t} + (\underline{V_\ell}\cdot\nabla)\underline{V_\ell} = k(\underline{V} - \underline{V_\ell}) \tag{3.9a}$$

$$\frac{dh_{\ell s}}{dt} = 0 \tag{3.10a}$$

$$h_{\ell s} = h_\ell + \frac{\gamma-1}{2}V_\ell^2 = h_{\ell s o} \tag{3.11a}$$

and now let us proceed by introducing small perturbations.

We will consider each of the dependent variables as the sum of a steady state space-variable and a time-dependent perturbation so small that terms higher than those linear in the perturbations can be neglected. Thus $\rho = \bar{\rho} + \rho'$, $p = \bar{p} + p'$ etc., where the superposed bar denotes steady state and the prime denotes small perturbation. The steady state equations and the equations linear in the perturbations follow directly.

Continuity

$$\nabla\cdot(\bar{\rho}\bar{\underline{V}}) = \bar{\phi} = -\nabla\cdot(\bar{\rho}_\ell \bar{\underline{V}}_\ell) \tag{3.2b}$$

$$\frac{\partial\rho'}{\partial t} + \nabla\cdot(\bar{\rho}\underline{V}' + \rho'\bar{\underline{V}}) = \phi' = -\frac{\partial\rho_\ell'}{\partial t} - \nabla\cdot(\bar{\rho}_\ell \underline{V}_\ell' + \rho_\ell'\bar{\underline{V}}_\ell) \tag{3.2c}$$

Momentum

$$\bar{\rho}\bar{\underline{V}}(\nabla\cdot\bar{\underline{V}}) + (\bar{\underline{V}}\cdot\nabla)\bar{\rho}\bar{\underline{V}} + \bar{\rho}_\ell\bar{\underline{V}}_\ell(\nabla\cdot\bar{\underline{V}}_\ell) + (\bar{\underline{V}}_\ell\cdot\nabla)\bar{\rho}_\ell\bar{\underline{V}}_\ell = -\frac{1}{\gamma}\nabla\bar{p} \tag{3.3b}$$

$$\frac{\partial}{\partial t}(\bar{\rho}\underline{V}' + \rho'\bar{\underline{V}} + \bar{\rho}_\ell\underline{V}_\ell' + \rho_\ell'\bar{\underline{V}}_\ell) + \bar{\rho}\bar{\underline{V}}(\nabla\cdot\underline{V}') \tag{3.3c}$$

$$+ (\bar{\rho}\underline{V}' + \rho'\bar{\underline{V}})\nabla\cdot\bar{\underline{V}} + (\bar{\underline{V}}\cdot\nabla)(\bar{\rho}\underline{V}' + \rho'\bar{\underline{V}}) + (\underline{V}'\cdot\nabla)(\bar{\rho}\bar{\underline{V}})$$

$$+ \bar{\rho}_\ell\bar{\underline{V}}_\ell(\nabla\cdot\underline{V}_\ell') + (\bar{\rho}_\ell\underline{V}_\ell' + \rho_\ell'\bar{\underline{V}}_\ell)(\nabla\cdot\bar{\underline{V}}_\ell) + (\bar{\underline{V}}_\ell\cdot\nabla)(\bar{\rho}_\ell\underline{V}_\ell' + \rho_\ell'\bar{\underline{V}}_\ell)$$

$$+ (\underline{V}_\ell'\cdot\nabla)(\bar{\rho}_\ell\bar{\underline{V}}_\ell) = -\frac{1}{\gamma}\nabla p'$$

Energy

$$(\bar{\rho}\bar{\underline{V}}\cdot\nabla)\bar{h}_s = -\bar{\phi}(\bar{h}_s - \bar{h}_{\ell s}) \tag{3.7b}$$

$$\bar{\rho}\frac{\partial h_s'}{\partial t} + \bar{\rho}\bar{\underline{V}}\cdot\nabla h_s' + (\bar{\rho}\underline{V}' + \rho'\bar{\underline{V}})\cdot\nabla\bar{h}_s \tag{3.7c}$$

$$= \frac{\gamma-1}{\gamma}\frac{\partial p'}{\partial t} - \bar{\phi}(h_s' - h_{\ell s}') - \phi'(\bar{h}_s - \bar{h}_{\ell s})$$

State

$$\bar{p} = \bar{\rho}\bar{T} \tag{3.8b}$$

$$\frac{p'}{\bar{p}} = \frac{\rho'}{\bar{\rho}} + \frac{T'}{\bar{T}} \tag{3.8c}$$

Droplet Dynamics

$$(\bar{\underline{V}}_\ell \cdot \nabla)\, \bar{\underline{V}}_\ell = k\,(\bar{\underline{V}} - \bar{\underline{V}}_\ell) \qquad 3.9b$$

$$\frac{\partial \underline{V}_\ell'}{\partial t} + (\bar{\underline{V}}_\ell \cdot \nabla)\, \underline{V}_\ell' + (\underline{V}_\ell' \cdot \nabla)\, \bar{\underline{V}}_\ell = k\,(\underline{V}' - \underline{V}_\ell') \qquad 3.9c$$

Heat Transfer

$$\bar{h}_{\ell s} = \bar{h}_\ell + \frac{\gamma-1}{2}\, \bar{V}_\ell^{\,2} = \bar{h}_{\ell s o} \qquad 3.11b$$

$$h_{\ell s}' = h_\ell' + (\gamma-1)\, \bar{\underline{V}}_\ell \cdot \underline{V}_\ell' = h_{\ell s o}' \qquad 3.11c$$

At this point, we have obtained a set of partial differential equations which govern the motion of the rocket fluid system during steady and unsteady operation. We are now interested in determining the stability of solutions with exponential time dependence, but first we proceed with the selection of an appropriate coordinate system.

4. Coordinate System

The natural coordinate system for rocket motors of conventional shape is a cylindrical coordinate system and hence the latter will be utilized here. Referring to Fig. 4.1, we see that the square of the elementary length is given by:

$$d\ell^{*2} = dz^{*2} + dr^{*2} + r^{*2}\, d\theta^{*2} \qquad 4.1$$

Non-dimensionalizing,

$$z = \frac{z^*}{r_c^*}, \qquad r = \frac{r^*}{r_c^*}, \qquad \theta^* = \theta \qquad 4.2$$

we obtain

$$d\ell^2 = dz^2 + dr^2 + r^2\, d\theta^2 \qquad 4.3$$

which enables us to expand the vector equations we have derived previously. In the ensuing analysis, it will be assumed that the steady state solution consists of one-dimensional flow, and then the solution in unsteady flow will consist of three dimensional perturbations superposed on the steady state solution. It follows that all partial derivatives of steady state quantities with respect to r or θ vanish, while partial derivatives of steady state quantities with respect to z become ordinary derivatives.

Letting the subscripts z, r and θ denote components of vectors in the respective directions, the continuity equations become:

$$\frac{d}{dz}(\bar{\rho}\,\bar{V}_z) = \bar{\phi} = -\frac{d}{dz}(\bar{\rho}_\ell\,\bar{V}_{\ell z}) \qquad 4.4$$

$$\frac{\partial \rho'}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial z}(r\bar{\rho}V_z') + \frac{\partial}{\partial r}(r\bar{\rho}V_r') + \frac{\partial}{\partial \theta}(\bar{\rho}V_\theta')\right] + \frac{\partial}{\partial z}(\rho'\bar{V}_z) \qquad 4.5$$

$$= \phi' = -\frac{\partial \rho_\ell'}{\partial t} - \frac{1}{r}\left[\frac{\partial}{\partial z}(r\bar{\rho}_\ell V_{\ell z}') + \frac{\partial}{\partial r}(r\bar{\rho}_\ell V_{\ell r}') + \frac{\partial}{\partial \theta}(\bar{\rho}_\ell V_{\ell\theta}')\right]$$

$$-\frac{\partial}{\partial z}(\rho_\ell'\bar{V}_{\ell z})$$

Proceeding, we obtain the equations for the conservation of momentum:

$$\frac{d}{dz}(\bar{\rho}\bar{V}_z^2) + \frac{d}{dz}(\bar{\rho}_\ell\bar{V}_{\ell z}^2) = -\frac{1}{\gamma}\frac{d\bar{p}}{dz} \qquad 4.6$$

<u>z Component of Momentum</u>

$$\frac{\partial}{\partial t}(\bar{\rho}V_z' + \rho'\bar{V}_z + \bar{\rho}_\ell V_{\ell z}' + \rho_\ell'\bar{V}_{\ell z}) + \bar{\rho}\bar{V}_z\left(\frac{\partial V_r'}{\partial r} + \frac{V_r'}{r} + \frac{1}{r}\frac{\partial V_\theta'}{\partial \theta}\right) \qquad 4.7$$

$$+ 2\bar{\rho}\bar{V}_z\frac{\partial V_z'}{\partial z} + 2\bar{\rho}V_z'\frac{d\bar{V}_z}{dz} + 2\bar{V}_z V_z'\frac{d\bar{\rho}}{dz} + 2\rho'\bar{V}_z\frac{d\bar{V}_z}{dz} + \bar{V}_z^2\frac{\partial \rho'}{\partial z}$$

$$+ \bar{\rho}_\ell\bar{V}_{\ell z}\left(\frac{\partial V_{\ell r}'}{\partial r} + \frac{V_{\ell r}'}{r} + \frac{1}{r}\frac{V_{\ell\theta}'}{\partial\theta}\right) + 2\bar{\rho}_\ell\bar{V}_{\ell z}\frac{\partial V_{\ell z}'}{\partial z} + 2\bar{\rho}_\ell V_{\ell z}'\frac{d\bar{V}_{\ell z}}{dz}$$

$$+ 2\bar{V}_{\ell z}V_{\ell z}'\frac{d\bar{\rho}_\ell}{dz} + 2\rho_\ell'\bar{V}_{\ell z}\frac{d\bar{V}_{\ell z}}{dz} + \bar{V}_{\ell z}^2\frac{\partial \rho_\ell'}{\partial z} = -\frac{1}{\gamma}\frac{\partial p'}{\partial z}$$

r Component of Momentum

$$\frac{\partial}{\partial t}(\bar{\rho} V_r' + \bar{\rho}_\ell V_{\ell r}') + \bar{\rho}\frac{d\bar{V}_z}{dz}V_r' + \bar{\rho}\bar{V}_z\frac{\partial V_r'}{\partial z} + \bar{V}_z V_r'\frac{d\bar{\rho}}{dz} \qquad 4.8$$

$$+ \bar{\rho}_\ell\frac{d\bar{V}_{\ell z}}{dz}V_{\ell r}' + \bar{\rho}_\ell\bar{V}_{\ell z}\frac{\partial V_{\ell r}'}{\partial z} + \bar{V}_{\ell z}V_{\ell r}'\frac{d\bar{\rho}_\ell}{dz} = -\frac{1}{\gamma}\frac{\partial p'}{\partial r}$$

θ Component of Momentum

$$\frac{\partial}{\partial t}(\bar{\rho} V_\theta' + \bar{\rho}_\ell V_{\ell\theta}') + \bar{\rho}\frac{d\bar{V}_z}{dz}V_\theta' + \bar{\rho}\bar{V}_z\frac{\partial V_\theta'}{\partial z} + \bar{V}_z V_\theta'\frac{d\bar{\rho}}{dz} \qquad 4.9$$

$$+ \bar{\rho}_\ell\frac{d\bar{V}_{\ell z}}{dz}V_{\ell\theta}' + \bar{\rho}_\ell\bar{V}_{\ell z}\frac{\partial V_{\ell\theta}'}{\partial z} + \bar{V}_{\ell z}V_{\ell\theta}'\frac{d\bar{\rho}_\ell}{dz} = -\frac{1}{\gamma}\frac{\partial p'}{r\partial\theta}$$

In considering the energy equation, an order of magnitude analysis shows that $\bar{h}_s$ is constant throughout the chamber to within terms of the order of the square of the local Mach number and hence from Eq. 3.7b it follows that $\bar{h}_s = \bar{h}_{\ell s}$. We will justify this order of magnitude analasis in another section. Therefore,

$$\bar{\rho}\frac{\partial h_s'}{\partial t} + \bar{\rho}\bar{V}_z\frac{\partial h_s'}{\partial z} = \frac{\gamma-1}{\gamma}\frac{\partial p'}{\partial t} - \bar{\phi}(h_s' - h_{\ell s}')$$

We may eliminate $\bar{\phi}$ by introducing Eq. 4.4 and we can eliminate h_s' as follows. Since $dh = dT$, we must also have $h' = T'$ and therefore $h_s' = T' + (\gamma-1)\,\bar{\underline{V}}\cdot\underline{V}' = T' + (\gamma-1)\bar{V}_z V_z'$. Introducing the equation of state, $T' = \bar{T}\left(\frac{p'}{\bar{p}} - \frac{\rho'}{\bar{\rho}}\right)$ we obtain for the energy equation:

$$\frac{\partial}{\partial t}\left[p' - \bar{T}\rho' + (\gamma-1)\bar{\rho}\bar{V}_z V_z'\right] + \frac{\partial}{\partial z}\left[\bar{V}_z\left\{p' - \bar{T}\rho' + (\gamma-1)\bar{\rho}\bar{V}_z V_z'\right\}\right] \qquad 4.10$$

$$= \frac{\gamma-1}{\gamma}\frac{\partial p'}{\partial t} + h_{ps}'\frac{d}{dz}(\bar{\rho}\bar{V}_z)$$

The droplet dynamics yield:

$$\bar{V}_{\ell z} \frac{d\bar{V}_{\ell z}}{dz} = k\left(\bar{V}_z - \bar{V}_{\ell z}\right) \quad 4.11$$

$$\frac{\partial V_{\ell z}'}{\partial t} + \bar{V}_{\ell z}\frac{\partial V_{\ell z}'}{\partial z} + V_{\ell z}'\frac{d\bar{V}_{\ell z}}{dz} = k\left(V_z' - V_{\ell z}'\right) \quad 4.12$$

$$\frac{\partial V_r'}{\partial t} + \bar{V}_{\ell z}\frac{\partial V_{\ell r}'}{\partial z} = k\left(V_r' - V_{\ell r}'\right) \quad 4.13$$

$$\frac{\partial V_{\ell\theta}'}{\partial t} + \bar{V}_{\ell z}\frac{\partial V_{\ell\theta}'}{\partial z} = k\left(V_\theta' - V_{\ell\theta}'\right) \quad 4.14$$

while the droplet stagnation enthalpy is simply

$$h_{\ell s}' = h_\ell' + (\gamma - 1)\bar{V}_{\ell z}V_{\ell z}' = h_{\ell s o}' \quad 4.15$$

5. Separation of the Variables

We may separate the variables in our system of linear partic dif-ferential equations by expressing each of the perturbations as the product of functions of the coordinates and an exponential time function. After ex-amining purely arbitrary functions, it is found that seperation may be achieved when we take:

$$\begin{aligned}
V_z' &= \nu_z(z)\,\Psi(r)\,\Phi(\theta)e^{st} \\
V_r' &= \nu_r(z)\,\frac{d\Psi(r)}{dr}\,\Phi(\theta)e^{st} \\
V_\theta' &= \nu_\theta(z)\,\frac{\Psi(r)}{r}\,\frac{d\Phi}{d\theta}(\theta)e^{st} \\
V_{\ell z}' &= \eta_z(z)\,\Psi(r)\,\Phi(\theta)e^{st} \\
V_{\ell r}' &= \eta_r(z)\,\frac{d\Psi(r)}{dr}\,\Phi(\theta)e^{st} \\
V_{\ell\theta}' &= \eta_\theta(z)\,\frac{\Psi(r)}{r}\,\frac{d\Phi}{d\theta}(\theta)e^{st} \\
\rho' &= \delta(z)\,\Psi(r)\,\Phi(\theta)e^{st} \\
\rho_\ell' &= \varsigma(z)\,\Psi(r)\,\Phi(\theta)e^{st} \\
p' &= \varphi(z)\,\Psi(r)\,\Phi(\theta)e^{st} \\
\phi' &= \frac{dq(z)}{dz}\,\Psi(r)\,\Phi(\theta)e^{st}
\end{aligned} \quad 5.1$$

Since the exponent $s = \Lambda + i\Omega$ is generally complex, the stability of any solution will depend on Λ . That is, the solution will be stable, neutral or unstable as Λ is less than, equal to, or exceeds, zero. We will determine the conditions yielding neutral stability, inasmuch as a knowledge of the stability boundary (Λ = 0) and the unstable side will be sufficient for our purposes.

We note that although the perturbations are given as complex quantities for the purpose of indicating phase relations, only the real parts have physical meaning. Before solving the perturbed equations we refer to Ref. 18 for the solutions to the steady state equations. Steady state continuity and momentum yield .

$$\bar{\rho}\,\bar{V}_z = \int_0^{\bar{z}} \bar{\phi}(\bar{z})\,d\bar{z} = \bar{w} = \bar{\rho}_{\ell i}\bar{V}_{\ell i} - \bar{\rho}_{\ell}\bar{V}_{\ell z} \qquad 5.2$$

where $\bar{\rho}_{\ell i}\bar{V}_{\ell i}$ represents the known propellant injection rate and $\bar{w}$ is the gas flow rate at any station, and

$$\bar{p} = 1 - \gamma\left(\bar{\rho}\,\bar{V}_{\ell z}^{2} + \bar{\rho}_{\ell}\bar{V}_{\ell z}^{2} - \bar{\rho}_{\ell o}\bar{V}_{\ell z_o}^{2}\right) \qquad 5.3$$

while we may also write

$$\bar{T} = 1 - \frac{\gamma - 1}{2}\,\bar{V}_z^{2} \qquad 5.4$$

Utilizing Eq. 3.8b we also obtain:

$$\bar{\rho} = \frac{\bar{p}}{1 - \frac{1}{2}(\gamma-1)\,\bar{V}_z^{2}} \qquad 5.5$$

Considering orders of magnitude, we note that the velocity increases from zero at the injector end to a maximum value connected with the Mach number at the entrance to the nozzle. A solution of the non-linear differential Equation 4.11 (see Fig. 4.2) shows that .... droplet velocity $\bar{V}_{\ell z}$ has the same order of magnitude as $\bar{V}_z$ and k . That is, $\bar{V}_z$, $\bar{V}_{\ell z}$ and k are each of the order of the Mach number. Furthermore, the deviations of $\bar{p}$, $\bar{\rho}$ and $\bar{T}$ from unity are of $O(M^2)$. Up to terms of this

order, we can therefore write:

$$\bar{p} = \bar{\rho} = \bar{T} = 1 \qquad 5.6$$

and hence there follows from Eq. 5.2.

$$\bar{\rho}_\ell \bar{V}_{\ell z} = \bar{V}_{z_e} - \bar{V}_z \qquad 5.7$$

since combustion is assumed complete at the exit of the chamber and

$\rho_{\ell i} \bar{V}_{\ell i} = \rho_e \bar{V}_{ze}$.

If we now take

$$\frac{d\bar{V}_z}{dz} \leqq O(1) \qquad \frac{1}{\varphi_0}\frac{d\nu_z}{dz} \leqq O(1) \qquad 5.8$$

$$\frac{d\bar{V}_{\ell z}}{dz} \leqq O(M) \qquad \frac{1}{\varphi_0}\frac{d\nu_r}{dz} \leqq O(1)$$

$$\frac{1}{\varphi_0}\frac{d\nu_\theta}{dz} \leqq O(1)$$

and if the injector does not respond to chamber pressure fluctuations (as in a study of intrinsic instability), then the droplet Equations 4.12, 4.13 and 4.14 yield simply:

$$\eta_z = \frac{k}{s}\nu_z \;; \quad \eta_r = \frac{k}{s}\nu_r \;; \quad \eta_\theta = \frac{k}{s}\nu_\theta \qquad 5.9$$

It will be shown, in our treatment of entropy wave instability that a more complicated result is obtained at lower frequencies, since the injection system can then respond to chamber pressure oscillations.

And now substituting from Eqs. 5.1 into the continuity Equation 4.5, we find for the left hand side:

$$\frac{s\delta + \frac{d}{dz}(\bar{\rho}\nu_z) + \frac{d}{dz}(\bar{V}_z\delta) - \frac{dq}{dz}}{\bar{\rho}\nu_r} = -\frac{\frac{d^2\psi}{dr^2}}{\psi} - \frac{\frac{d\psi}{dr}}{r\psi} - \frac{\nu_\theta}{\nu_r}\frac{\frac{d^2\Phi}{d\theta^2}}{r^2\Phi^2} \qquad 5.10$$

where the variables are not separated as yet. We will return to this equation presently.

Upon introduc Eqs. 5.1 and 5.9 into the equation for the

component of momentum, Eq. 4.7, and rearranging terms, we obtain:

$$\frac{d}{dz}\left[2\left(\bar\rho\bar V_z+\frac{k}{s}\bar\rho_\ell\bar V_{\ell z}\right)\nu_z+\bar V_z^2\delta+\bar V_{\ell z}^2\varsigma\right]+s\left[\left(\bar\rho+\frac{k}{s}\bar\rho_\ell\right)\nu_z+\bar V_z\delta+\bar V_{\ell z}\varsigma\right] \qquad 5.11$$

$$+\frac{1}{\gamma}\frac{d\varphi}{dz}=-\left[1+\frac{k}{s}\frac{\bar\rho_\ell\bar V_{\ell z}}{\bar\rho\bar V_z}\right]\bar\rho\bar V_z\nu_r\left[\frac{\frac{d^2\Psi}{dr^2}}{\Psi}+\frac{\frac{d\Psi}{dr}}{r\Psi}+\frac{\nu_\theta}{\nu_r}\frac{\frac{d^2\Phi}{d\theta^2}}{r^2\Phi}\right]$$

The r and θ components of momentum, Eqs. 4.8 and 4.9 yield:

$$\frac{d}{dz}\left[\left(\bar\rho\bar V_z+\frac{k}{s}\bar\rho_\ell\bar V_{\ell z}\right)\nu_r\right]+s\left[\bar\rho+\frac{k}{s}\bar\rho_\ell\right]\nu_r=-\frac{\varphi}{\gamma} \qquad 5.12$$

and

$$\frac{d}{dz}\left[\left(\bar\rho\bar V_z+\frac{k}{s}\bar\rho_\ell\bar V_{\ell z}\right)\nu_\theta\right]+s\left[\bar\rho+\frac{k}{s}\bar\rho_\ell\right]\nu_\theta=-\frac{\varphi}{\gamma} \qquad 5.13$$

Expanding Eqs. 5.12 and 5.13 and then subtracting one from the other we obtain:

$$\left[\bar\rho\bar V_z+\frac{k}{s}\bar\rho_\ell\bar V_{\ell z}\right]\left(\frac{d\nu_r}{dz}-\frac{d\nu_\theta}{dz}\right)+\left[\frac{d}{dz}\left(\bar\rho\bar V_z+\frac{k}{s}\bar\rho_\ell\bar V_{\ell z}\right)+s\left(\bar\rho+\frac{k}{s}\bar\rho_\ell\right)\right](\nu_r-\nu_\theta)=0 \qquad 5.14$$

This has the solution:

$$\nu_r-\nu_\theta=C_1\exp\left\{-\int^z\frac{\frac{d}{dz}\left(\bar\rho\bar V_z+\frac{k}{s}\bar\rho_\ell\bar V_{\ell z}\right)+s\left(\bar\rho+\frac{k}{s}\bar\rho_\ell\right)}{\bar\rho\bar V_z+\frac{k}{s}\bar\rho_\ell\bar V_{\ell z}}\,dz\right\} \qquad 5.15$$

and hence unless C_1 is taken identically zero, separation of the variables is precluded.

It is of some interest to see what happens to the vorticity as a consequence of this last result. By definition, the vorticity is given by:

$$\mathrm{rot}\,\underline{V}'=\nabla\times\underline{V}'=\frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta')-\frac{\partial}{\partial\theta}(V_r')\right]\underline{u_z} \qquad 5.16$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial\theta}(V_z')-\frac{\partial}{\partial z}(rV_\theta')\right]\underline{u_r}+\frac{1}{r}\left[\frac{\partial}{\partial z}(V_r')-\frac{\partial}{\partial r}(V_z')\right]\underline{u_\theta}$$

Substituting from Eq. 5.1 we obtain:

$$(\mathrm{rot}\ \underset{\to}{v}')_z = (\nu_\theta - \nu_r)\ \frac{d\Psi}{dr}\ \frac{1}{r}\ \frac{d\Phi}{d\theta}\ e^{st} = 0 \qquad 5.17$$

$$(\mathrm{rot}\ \underset{\to}{v}')_r = \left(\nu_z - \frac{d\nu_\theta}{dz}\right) \frac{\Psi}{r}\ \frac{d\Phi}{d\theta}\ e^{st}$$

$$(\mathrm{rot}\ \underset{\to}{v}')_\theta = \left(\frac{d\nu_r}{dz} - \nu_z\right) \frac{d\Psi}{dr}\ \Phi\ e^{st}$$

and thus we see that the condition that enables us to separate the variables, namely $\nu_\theta = \nu_r$, also causes the axial component of the vorticity to vanish.

Returning to continuity, Eq. 5.10, and introducing $\nu_\theta = \nu_r$, we find:

$$\frac{s\delta + \frac{d}{dz}(\bar{\rho}\nu_z) + \frac{d}{dz}(\bar{V}_z\delta) - \frac{dq}{dz}}{\bar{\rho}\nu_r} = -\frac{\frac{d^2\Psi}{dr^2}}{\Psi} - \frac{\frac{d\Psi}{dr}}{r\Psi} - \frac{\frac{d^2\Phi}{d\theta^2}}{r^2\Phi} = S_{nh}^2 \qquad 5.18$$

where S_{nh}^2 is the separation constant, and we also obtain

$$r^2\frac{\frac{d^2\Psi}{dr^2}}{\Psi} + r\frac{\frac{d\Psi}{dr}}{\Psi} + S_{nh}^2 r^2 = -\frac{\frac{d^2\Phi}{d\theta^2}}{\Phi} = n^2 \qquad 5.19$$

where n^2 is the second separation constant. We have thus obtained the set:

$$r^2\frac{d^2\Psi}{dr^2} + r\frac{d\Psi}{dr} + (r^2 S_{nh}^2 - n^2)\Psi = 0 \qquad 5.20$$

$$\frac{d^2\Phi}{dn^2} + n^2\Phi = 0 \qquad 5.21$$

Equation 5.20 is recognized as a Bessel equation so that the solutions become respectively,

$$\Psi = \begin{cases} J_n(S_{nh}r) \\ Y_n(S_{nh}r) \end{cases} \qquad 5.22$$

$$\Phi = \begin{cases} \cos n\theta \\ \sin n\theta \end{cases} \qquad 5.23$$

No constants need appear above as they may be conveniently absorbed into the z-dependent factors. The actual solution is obtained by summing over all combinations of products of solutions. We must satisfy certain conditions imposed on V_r'. First, V_r' must not become infinite at $r = 0$, and hence we must discard the second Bessel function. Further, V_r' must vanish at the cylinder wall where $r = 1$, and hence Eq. 5.1 implies that:

$$\left.\frac{d\psi}{dr}\right)_{r=1} = \frac{d\psi}{dr}(s_{nh}) = 0 \qquad 5.24$$

Since there are a doubly infinite number of solutions to this equation, we let s_{nh} represent the h^{th} zero of the derivative of the Bessel function of order n, so that both subscripts of s_{nh} are used as indices to indicate which of the solutions interest us. The first nine zeros are:

$s_{01} = 0$		$s_{02} = 3.8317$		$s_{03} = 7.0156$		5.25
$s_{11} = 1.8413$		$s_{12} = 5.3313$		$s_{13} = 8.5263$		
$s_{21} = 3.0543$		$s_{22} = 6.7060$		$s_{23} = 9.9695$		

The system of equations which must be solved simultaneously now becomes:

Continuity

$$s\delta + \frac{d}{dz}(\bar{\rho}\nu_z + \bar{V}_z\delta) - s^2_{nh}\bar{\rho}\nu_r = \frac{dq}{dz} = \qquad 5.26$$

$$-s\varsigma - \frac{d}{dz}\left(\frac{k}{s}\bar{\rho}_\ell\nu_z + \bar{V}_{\ell z}\varsigma\right) + s^2_{nh}\frac{k}{s}\bar{\rho}_\ell\nu_r \quad .$$

Momentum z

$$s(\bar{\rho}\nu_z + \bar{V}_z\delta) + \frac{d}{dz}(2\bar{\rho}\bar{V}_z\nu_z + \bar{V}_z^2\delta) = -\frac{1}{\gamma}\frac{d\varphi}{dz} \qquad 5.27$$

$$+ s^2_{nh}\left[1 + \frac{k}{s}\frac{\bar{\rho}_\ell}{\bar{\rho}}\frac{\bar{V}_{\ell z}}{\bar{V}_z}\right]\bar{\rho}\bar{V}_z\nu_r - s\left(\frac{k}{s}\bar{\rho}_\ell\nu_z + \bar{V}_{\ell z}^2\varsigma\right)$$

$$-\frac{d}{dz}\left(2\frac{k}{s}\bar{\rho}_\ell\bar{V}_{\ell z}\nu_z + \bar{V}_{\ell z}^2\varsigma\right)$$

Momentum r

$$s\bar{\rho}\nu_r + \frac{d}{dz}(\bar{\rho}\bar{V}_z\nu_r) = -\frac{\varphi}{\gamma} - s\frac{k}{s}\bar{\rho}_\ell\nu_r - \frac{k}{s}\frac{d}{dz}(\bar{\rho}_\ell\bar{V}_{\ell z}\nu_r) \qquad 5.28$$

Energy

$$\frac{d}{dz}\left[\bar{V}_z\left\{\varphi - \bar{T}\delta + (\gamma-1)\bar{\rho}\bar{V}_z\nu_z\right\}\right] + s\left[\frac{\varphi}{\gamma} - \bar{T}\delta + (\gamma-1)\bar{\rho}\bar{V}_z\nu_z\right] = 0 \qquad 5.29$$

Let us review the initial conditions at the injector face for the case of intrinsic instability studies. At $z = 0$,

$$\begin{aligned} V_z'(0,r,\theta,t) &= 0 \\ \rho_\ell'(0,r,\theta,t) &= 0 \\ \phi'(0,r,\theta,t) &= 0 \\ \bar{V}_z(0,r,\theta,t) &= 0 \end{aligned} \qquad 5.30$$

since at the injector face there is no production of gas, and since the liquid is incompressible there can be no variation in propellant density. Equivalently, we have

$$\nu_z(0) = \eta_z(0) = \zeta(0) = \frac{d\zeta}{dz}(0) = \bar{V}_z(0) = 0 \qquad 5.31$$

while $\varphi(0) = \varphi_0$, $\nu_r(0) = \nu_{r0}$, and $\delta(0) = \delta_0$.

We will proceed to solve our system of equations by an iteration scheme, (see Ref. 18). Combining continuity and energy, Eqs, 5.26 and 5.29, and letting

$$\begin{aligned} X(z) &= (\gamma-1)\bar{\rho}\bar{V}_z\frac{\nu_z}{\varphi_0} + (1-\bar{T})\frac{\delta}{\varphi_0} \\ Y(z) &= \frac{\zeta}{\varphi_0} - \bar{V}_z\frac{\varphi}{\varphi_0} + (1-\bar{\rho})\frac{\nu_z}{\varphi_0} - \bar{V}_z(1-\bar{T})\frac{\delta}{\varphi_0} - (\gamma-1)\bar{\rho}\bar{V}_z^2\frac{\nu_z}{\varphi_0} \\ V(z) &= (1-\bar{\rho})\frac{\nu_r}{\varphi_0} \end{aligned} \qquad 5.32$$

we obtain:

$$\frac{d}{dz}\left(\frac{\nu_z}{\varphi_0}\right) + s\left(\frac{\varphi}{\gamma\varphi_0}\right) - s^2 nh\frac{\nu_r}{\varphi_0} = -sX + \frac{dY}{dz} - s^2 nhV \qquad 5.33$$

Rearranging Eq. 5.27 we obtain:

$$\frac{d}{dz}\left(\frac{\varphi}{\gamma\varphi_0}\right) + s\,\frac{\nu_z}{\varphi_0} = -s\,Z - \frac{dW}{dz} + s_{nh}^2\,U \qquad 5.34$$

where we have taken:

$$\begin{aligned} Z(z) &= \bar{V}_z\,\frac{\delta}{\varphi_0} - (1-\bar{\rho})\,\frac{\nu_z}{\varphi_0} + \frac{k}{s}\,\bar{\rho}_e\,\frac{\nu_z}{\varphi_0} + \bar{V}_{ez}\,\frac{\varsigma}{\varphi_0} \\ W(z) &= 2\bar{\rho}\,\bar{V}_z\,\frac{\nu_z}{\varphi_0} + 2\,\frac{k}{s}\,\bar{\rho}_e\,\bar{V}_{ez}\,\frac{\nu_z}{\varphi_0} + \bar{V}_z^{\,2}\,\frac{\delta}{\varphi_0} + \bar{V}_{ez}^{\,2}\,\frac{\varsigma}{\varphi_0} \\ U(z) &= \bar{\rho}\,\bar{V}_z\,\frac{\nu_r}{\varphi_0} + \frac{k}{s}\,\bar{\rho}_e\,\bar{V}_{ez}\,\frac{\nu_r}{\varphi_0} \end{aligned} \qquad 5.35$$

Our last equation, Eq. 5.28 becomes:

$$s\,\frac{\nu_r}{\varphi_0} + \frac{\varphi}{\gamma\varphi_0} = -sM - \frac{dU}{dz} \qquad 5.36$$

upon taking

$$M(z) = \frac{k}{s}\,\bar{\rho}_e\,\frac{\nu_r}{\varphi_0} - (1-\bar{\rho})\,\frac{\nu_r}{\varphi_0} \qquad 5.37$$

For ready reference, we repeat the new simultaneous set below:

$$\begin{aligned} \frac{d}{dz}\left(\frac{\nu_z}{\varphi_0}\right) + s\left(\frac{\varphi}{\gamma\varphi_0}\right) - s_{nh}^2\left(\frac{\nu_r}{\varphi_0}\right) &= -sX + \frac{dY}{dz} - s_{nh}^2\,V \\ \frac{d}{dz}\left(\frac{\varphi}{\gamma\varphi_0}\right) + s\left(\frac{\nu_z}{\varphi_0}\right) &= -sZ - \frac{dW}{dz} + s_{nh}^2\,U \\ s\left(\frac{\nu_r}{\varphi_0}\right) + \left(\frac{\varphi}{\gamma\varphi_0}\right) &= -sM - \frac{dU}{dz} \end{aligned} \qquad 5.38$$

We note immediately that if $s_{nh} = 0$, which implies $V_r' = V_\theta' = 0$, then the treatment reduces to a purely longitudinal mode of oscillation, which has already been analyzed by Crocco and Cheng in Ref. 18.

Let us proceed with the solution of Eqs. 5.38. Rearranging the third equation and substituting it into the first:

$$\frac{d}{dz}\left(\frac{\nu_z}{\varphi_0}\right) + \left(s + \frac{s_{nh}^2}{s}\right)\left(\frac{\varphi}{\gamma\varphi_0}\right) = -sX + \frac{dY}{dz} - s_{nh}^2\left(V + M + \frac{1}{s}\,\frac{dU}{dz}\right)$$

Adding and subtracting terms in this equation and also in the second Equation 5.38 above, we derive:

$$\frac{d}{dz}\left(\frac{\nu_z}{\varphi_0} - Y + \frac{s^2_{nh}}{s}U\right) + \left(s + \frac{s^2_{nh}}{s}\right)\left(\frac{\varphi}{\gamma\varphi_0} + W\right) = s(W - X) - s^2_{nh}\left(V + M - \frac{W}{s}\right) \qquad 5.39$$

$$\frac{d}{dz}\left(\frac{\varphi}{\gamma\varphi_0} + W\right) + s\left(\frac{\nu_z}{\varphi_0} - Y + \frac{s^2_{nh}}{s}U\right) = -s(Y + Z) + 2s^2_{nh}U \qquad 5.40$$

For convenience we let:

$$D(z) = \left(\frac{\nu_z}{\varphi_0} - Y + \frac{s^2_{nh}}{s}U\right)$$
$$B(z) = \left(\frac{\varphi}{\gamma\varphi_0} + W\right)$$
$$F(z) = Y + Z \qquad 5.41$$
$$E(z) = W - X$$
$$G(z) = V + M - \frac{W}{s}$$

and then Eqs. 5.39 and 5.40 become:

$$\frac{dD}{dz} + \left(s + \frac{s^2_{nh}}{s}\right)B = sE - s^2_{nh}G \qquad 5.42$$

$$\frac{dB}{dz} + sD = -sF + 2s^2_{nh}U \qquad 5.43$$

We may eliminate B by differentiating Eq. 5.42 and combining it with Eq. 5.53:

$$\frac{d^2D}{dz^2} - (s^2 + s^2_{nh})D = h(z) \qquad 5.44$$

where

$$h(z) = s\frac{dE}{dz} + s^2F + s^2_{nh}\left[F - \frac{dG}{dz} - 2\left(s + \frac{s^2_{nh}}{s}\right)U\right] \qquad 5.45$$

Since the right hand side of Eq. 5.44 is not given explicitly, we will use the method of variation of parameters to solve this ordinary differential equation. We obtain:

$$D(z) = -C_1\cosh\sqrt{s^2+s^2_{nh}}\,z - C_2\sinh\sqrt{s^2+s^2_{nh}}\,z + \frac{\int_0^z h(z')\sinh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'}{\sqrt{s^2+s^2_{nh}}} \qquad 5.46$$

We can now substitute back for $h(z')$ from Eq. 5.45. Then integrating by parts and combining terms, where the appropriate initial conditions are obtained from Eq. 5.31 i.e., $E(0)=0$ and $G(0) = \frac{k}{s}\bar{\rho}_{l0} \frac{v_{r0}}{\varphi_0}$, there is obtained:

$$D(z) = \frac{v_z}{\varphi_0} - Y + \frac{s^2 nh}{s} U = -C_1 \cosh\sqrt{s^2+s^2nh}\, z - C_2 \sinh\sqrt{s^2+s^2nh}\, z \qquad 5.47$$

$$+ s\int_0^z E(z') \cosh\sqrt{s^2+s^2nh}\,(z-z')\,dz'$$

$$+ \sqrt{s^2+s^2nh}\int_0^z F(z') \sinh\sqrt{s^2+s^2nh}\,(z-z')\,dz'$$

$$- s^2nh\int_0^z G(z') \cosh\sqrt{s^2+s^2nh}\,(z-z')\,dz'$$

$$- \sqrt{s^2+s^2nh}\left(\frac{s^2nh}{s}\right)\int_0^z U(z') \sinh\sqrt{s^2+s^2nh}\,(z-z')\,dz'$$

Noting that

$$\frac{d}{dz}\int_0^z f(z,z')\,dz = \int_0^z \frac{\partial f}{\partial z}(z,z')\,dz' + f(z,z)$$

we may differentiate Eq. 5.47 and substitute back into Eq. 5.42 to obtain B.

$$B(z) = \frac{\varphi}{\gamma\varphi_0} + W = \frac{s}{\sqrt{s^2+s^2nh}}\left[C_1 \sinh\sqrt{s^2+s^2nh}\, z + C_2 \cosh\sqrt{s^2+s^2nh}\, z\right] \qquad 5.48$$

$$- \frac{s^2}{\sqrt{s^2+s^2nh}}\int_0^z E(z') \sinh\sqrt{s^2+s^2nh}\,(z-z')\,dz'$$

$$- s\int_0^z F(z') \cosh\sqrt{s^2+s^2nh}\,(z-z')\,dz'$$

$$+ \frac{s^2nh}{\sqrt{s^2+s^2nh}}\int_0^z G(z') \sinh\sqrt{s^2+s^2nh}\,(z-z')\,dz'$$

$$+ 2s^2nh\int_0^z U(z') \cosh\sqrt{s^2+s^2nh}\,(z-z')\,dz'$$

We can now uniquely determine the constants C_1 and C_2 by introducing the conditions at $z = 0$. From Eq. 5.31 we have:

$$Y(0) = W(0) = 0 \qquad 5.49$$

$$U(0) = \frac{k}{s}\bar{\rho}_{l0}\bar{V}_{lz0}\frac{v_{r0}}{\varphi_0}$$

and hence simultaneous solution of Eqs. 5.47 and 5.48 at $z = 0$ yields:

$$C_1 = -\frac{s^2_{nh}}{s^2} k \bar{\rho}_{\ell 0} \bar{V}_{\ell z0} \frac{\nu_{r0}}{\varphi_0} \qquad 5.50$$

$$C_2 = \sqrt{s^2 + s^2_{nh}} \; \frac{1}{\gamma s}$$

and now substituting these results into Eqs. 5.47 and 5.48, we finally obtain:

$$\frac{\varphi}{\varphi_0} = \cosh\sqrt{s^2+s^2_{nh}}\, z - \frac{s^2_{nh}}{\sqrt{s^2+s^2_{nh}}} \frac{\gamma}{s} k \bar{\rho}_{\ell 0} \bar{V}_{\ell z0} \frac{\nu_{r0}}{\varphi_0} \sinh\sqrt{s^2+s^2_{nh}}\, z \qquad 5.51$$
$$- \frac{s^2}{\sqrt{s^2+s^2_{nh}}} \int_0^z \gamma E(z') \sinh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$
$$- s \int_0^z \gamma F(z') \cosh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$
$$+ \frac{s^2_{nh}}{\sqrt{s^2+s^2_{nh}}} \int_0^z \gamma G(z') \sinh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$
$$+ 2 s^2_{nh} \int_0^z \gamma U(z') \cosh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$

$$\frac{\gamma \nu_z}{\varphi_0} = -\frac{\sqrt{s^2+s^2_{nh}}}{s} \sinh\sqrt{s^2+s^2_{nh}}\, z + \frac{s^2_{nh}}{s} \gamma k \bar{\rho}_{\ell 0} \bar{V}_{\ell z0} \frac{\nu_{r0}}{\varphi_0} \cosh\sqrt{s^2+s^2_{nh}}\, z \qquad 5.52$$
$$+ \gamma Y - \frac{s^2_{nh}}{s} \gamma U + s \int_0^z \gamma E(z') \cosh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$
$$+ \sqrt{s^2+s^2_{nh}} \int_0^z \gamma F(z') \sinh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$
$$- s^2_{nh} \int_0^z \gamma G(z') \cosh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$
$$- 2\sqrt{s^2+s^2_{nh}} \left(\frac{s^2_{nh}}{s}\right) \int_0^z \gamma U(z') \sinh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$

We may also obtain a solution for $\frac{\gamma \nu_r}{\varphi_0}$ by introducing Eq. 5.51 into the third Eq. 5.38 as follows:

$$\frac{\gamma \nu_r}{\varphi_0} = -\frac{1}{s} \cosh\sqrt{s^2+s^2_{nh}}\, z + \frac{s^2_{nh}}{\sqrt{s^2+s^2_{nh}}} \frac{\gamma}{s^2} k \bar{\rho}_{\ell 0} \bar{V}_{\ell z0} \frac{\nu_{r0}}{\varphi_0} \sinh\sqrt{s^2+s^2_{nh}}\, z \qquad 5.53$$
$$- \gamma M - \frac{\gamma}{s} \frac{dU}{dz} + \frac{s}{\sqrt{s^2+s^2_{nh}}} \int_0^z \gamma E(z') \sinh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$
$$+ \int_0^z \gamma F(z') \cosh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$
$$- \frac{s^2_{nh}}{\sqrt{s^2+s^2_{nh}}} \frac{1}{s} \int_0^z \gamma G(z') \sinh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$
$$- 2 \frac{s^2_{nh}}{s} \int_0^z \gamma U(z') \cosh\sqrt{s^2+s^2_{nh}}\,(z-z')\,dz'$$

It is clear that the order of magnitude of the individual terms comprising Eqs. 5.51, 5.52 and 5.53 depends on the order of magnitude of S and S_{nh} .

We are also interested in determining the entropy variation in the gas, since a knowledge of this perturbation will be necessary in applying the boundary condition at the entrance of the exhaust nozzle. In dimensional form we write:

$$T^* dS^* = dh^* - \frac{1}{\rho^*} dp^* \tag{5.54}$$

which may be non-dimensionalized to yield:

$$T dS = dh - \frac{\gamma-1}{\gamma} \frac{dp}{\rho} \tag{5.55}$$

and since $dh = dT$, this is equivalent to:

$$\overline{T} S' = T' - \frac{\gamma-1}{\gamma} \frac{p'}{\overline{\rho}} \tag{5.56}$$

and now introducing the equation of state 3.8c, this becomes:

$$S' = \frac{1}{\gamma} \frac{p'}{\overline{p}} - \rho' \frac{\overline{T}}{\overline{p}} \tag{5.57}$$

In order to separate the variables in this equation, we take:

$$S' = \epsilon(z) \Psi(r) \Phi(\theta) e^{st} \tag{5.58}$$

and then introducing Eqs. 5.1 and 5.58, Eq. 5.57 becomes:

$$\overline{p} \frac{\epsilon}{\varphi_0} = \frac{1}{\gamma} \frac{\varphi}{\varphi_0} - \frac{\delta}{\varphi_0} \overline{T} \tag{5.59}$$

And now, rearranging the energy equation, Eq. 5.29, we have:

$$\frac{d}{dz}\left[\overline{V}_z\left\{\frac{\varphi}{\varphi_0} - \frac{\delta}{\varphi_0}\overline{T} + (\gamma-1)\overline{\rho}\,\overline{V}_z\frac{\nu_z}{\varphi_0}\right\}\right] + \frac{s}{\overline{V}_z}\left[\overline{V}_z\left\{\frac{\varphi}{\varphi_0} - \frac{\delta}{\varphi_0}\overline{T} + (\gamma-1)\overline{\rho}\,\overline{V}_z\frac{\nu_z}{\varphi_0}\right\}\right] = \frac{\gamma-1}{\gamma} s \frac{\varphi}{\varphi_0} \tag{5.60}$$

This linear differential equation has the solution:

$$\overline{V}_z\left\{\frac{\varphi}{\varphi_0} - \frac{\delta}{\varphi_0}\overline{T} + (\gamma-1)\overline{\rho}\,\overline{V}_z\frac{\nu_z}{\varphi_0}\right\} = e^{-s\int_0^z \frac{dz'}{\overline{V}_z(z')}}\left[\int_0^z s\,\frac{\gamma-1}{\gamma}\,\frac{\varphi(z')}{\varphi_0}\, e^{s\int_0^{z'}\frac{dz''}{\overline{V}_z(z'')}} dz' + \text{const.}\right] \tag{5.61}$$

where the constant must be taken zero, since the left hand side vanishes

at $z = 0$. Rearranging,

$$\frac{\varphi}{\varphi_0} - \frac{\delta}{\varphi_0}\bar{T} = -(\gamma-1)\bar{\rho}\bar{V}_z\frac{\nu_z}{\varphi_0} + \frac{s}{\bar{V}_z}\frac{\gamma-1}{\gamma}\int_0^z \frac{\varphi(z')}{\varphi_0}\, e^{s\int_z^{z'}\frac{dz''}{\bar{V}_z(z'')}}\, dz' \qquad 5.62$$

which may be integrated by parts to yield:

$$\frac{\varphi}{\varphi_0} - \frac{\delta}{\varphi_0}\bar{T} = -\frac{\gamma-1}{\gamma}\bar{\rho}\bar{V}_z\frac{\gamma\nu_z}{\varphi_0} + \frac{\gamma-1}{\gamma}\frac{\varphi}{\varphi_0} - \frac{\gamma-1}{\gamma}\frac{1}{\bar{V}_z}\int_0^z e^{s\int_z^{z'}\frac{dz''}{\bar{V}_z(z'')}}\, d\left[\bar{V}_z(z')\frac{\varphi(z')}{\varphi_0}\right] \qquad 5.63$$

Combining Eqs. 5.59 and 5.63 we have:

$$\bar{p}\frac{\gamma\epsilon}{\varphi_0} = -(\gamma-1)\left\{\bar{\rho}\bar{V}_z\frac{\gamma\nu_z}{\varphi_0} + \frac{1}{\bar{V}_z}\int_0^z e^{s\int_z^{z'}\frac{dz''}{\bar{V}_z(z'')}}\, d\left[\bar{V}_z(z')\frac{\varphi(z')}{\varphi_0}\right]\right\} \qquad 5.64$$

and now expanding the term in brackets, we obtain finally,

$$\begin{aligned}\bar{p}\frac{\gamma\epsilon}{\varphi_0} = -(\gamma-1)\Bigg\{&\bar{\rho}\bar{V}_z\frac{\gamma\nu_z}{\varphi_0} + \frac{1}{\bar{V}_z}\int_0^z e^{s\int_z^{z'}\frac{dz''}{\bar{V}_z(z'')}}\,\frac{\varphi(z')}{\varphi_0}\,\frac{d\bar{V}_z}{dz'}(z')\, dz' \\ &+ \frac{1}{\bar{V}_z}\int_0^z e^{s\int_z^{z'}\frac{dz''}{\bar{V}_z(z'')}}\,\bar{V}_z(z')\,\frac{d}{dz'}\left(\frac{\varphi(z')}{\varphi_0}\right) dz'\Bigg\}\end{aligned} \qquad 5.65$$

6. <u>The Burning Rate Perturbation</u>

In Section 3, it was pointed out that a relation between the burning rate and the other quantities would be required. This relationship will be derived below.

We recall that a conditioning process takes place during the time lag, and hence its duration is a function of the physico-chemical processes taking place during the time interval from the instant of injection to the instant that combustion of the given propellant is initiated. Since the steady state combustion rate has an arbitrary axial distribution, the total time lag will in general be different for different propellant elements.

Following Crocco and Cheng, the total time lag will be taken as the

sum of a space varying insensitive part $\bar{\tau}_i$, and a time and space varying sensitive part τ .

$$\tau_t(z,r,\theta,t) = \bar{\tau}_i(z) + \tau(z,r,\theta,t) \qquad 6.1$$

where τ is a function of the interaction index characteristic of the propellant combination. In the absence of mixture ratio variations, the rates may be related to the pressure by taking

$$f(z,r,\theta,t) = \bar{f}(z,r,\theta)\left[1 + n\,\frac{p'(z,r,\theta,t)}{\bar{p}}\right] \qquad 6.2$$

where f is the overall rate of the conditioning processes. Now, introducing Crocco's definition of the sensitive time lag, we have:

$$\int_{t-\tau}^{t} f(t')\,dt' = E_a = \bar{E}_a \qquad 6.3$$

where E_a represents the quantity of energy required to initiate burning at station z , r , θ at time t , and the integral must be evaluated following the motion of the particle. Equation 6.3 may be rewritten in the following equivalent forms.

$$\int_{t-\tau(z,r,\theta,t)}^{t} f\left[z'(t'),r'(t'),\theta'(t'),t'\right]dt' = E_a = \bar{E}_a \qquad 6.4$$

$$\int_{\xi,K,\chi}^{z,r,\theta} f\left[z',r',\theta',t'(z',r',\theta')\right]\frac{d\underline{\ell}}{\underline{V}_\ell} = E_a = \bar{E}_a \qquad 6.5$$

where $d\underline{\ell}$ denotes the particle path, and ξ , K , χ defines the location where the particle enters the sensitive phase. We note that according to the definition of the Lagrangian derivative, we may write, upon introducing Eq. 6.1:

$$V_{\ell z}(z,r,\theta,t) = \frac{dz}{dt} = \bar{V}_{\ell z}(z) + \eta_z(z)\,\Psi(r)\,\Phi(\theta)\,e^{st} \qquad 6.6$$

$$V_{\ell r}(z,r,\theta,t) = \frac{dr}{dt} = \eta_r(z)\,\frac{d\Psi(r)}{dr}\,\Phi(\theta)\,e^{st}$$

$$V_{\ell \theta}(z,r,\theta,t) = \frac{r\,d\theta}{dt} = \eta_\theta(z)\,\frac{\Psi(r)}{r}\,\frac{d\Phi(\theta)}{d\theta}\,e^{st}$$

Introducing Eqs. 6.2 and 6.6 into Eqs. 6.4 and 6.5, we have:

$$\int_{t-\tau}^{t} \bar{f}\left[z'(t'),r'(t'),\theta'(t'),t'\right]\left\{1 + n\,\frac{p'\left[z'(t'),r'(t'),\theta'(t'),t'\right]}{\bar{p}}\right\}dt' = E_a = \bar{E}_a(\bar{z}) \qquad 6.7$$

$$\int_{\xi}^{z} \frac{\bar{f}\left[z',r'(z'),\theta'(z'),t'(z')\right]\left\{1+n\,\frac{p'\left[z',r'(z'),\theta'(z'),t'(z')\right]}{\bar{p}}\right\}dz'}{\bar{V}_{\ell z}(z') + \eta_z(z')\,\Psi\left[r'(z')\right]\Phi\left[\theta'(z')\right]e^{st'(z')}} = E_a = \bar{E}_a(\bar{z}) \qquad 6.8$$

where $t'(z')$ is given by:

$$t'(z') = t(z,r,\theta) - \int_{z'}^{z} \frac{dz''}{\bar{V}_{\ell z}(z'') + V'_{\ell z}\left[z'',t''(z'')\right]} \qquad 6.9$$

Since the steady state solution is one-dimensional, we have:

$$\int_{t-\bar{\tau}(\bar{z})}^{t} \bar{f}\left[z'(t'),t'\right]dt' = \bar{E}_a(\bar{z}) \qquad 6.10$$

and furthermore, since in the steady state there is a negligible spatial non-uniformity, we also have $\bar{f}\left[z'(t'),t'\right] \cong \bar{f} = \text{const.}$ Hence Eq. 6.10 becomes:

$$\bar{f}\cdot\bar{\tau}(\bar{z}) = \bar{E}_a(\bar{z}) \qquad 6.11$$

and now Eq. 6.7 may be written:

$$\int_{t-\tau(z,r,\theta,t)}^{t} \bar{f}\left\{1+n\frac{p'}{\bar{p}}[z'(t'),r'(t'),\theta'(t'),t']\right\}dt' = \bar{E}_{\alpha}(\bar{z}) \qquad 6.12$$

Noting that $\tau(z,r,\theta,t) = \bar{\tau}(\bar{z}) + \tau'(z,r,\theta,t)$ we may neglect second order terms in Eq. 6.12 to obtain:

$$\begin{aligned}\tau'(z,r,\theta,t) &= \tau(z,r,\theta,t) - \bar{\tau}(\bar{z}) \\ &= -\int_{t-\tau(z,r,\theta,t)}^{t} n\frac{p'}{\bar{p}}[z'(t'),r'(t'),\theta'(t'),t']dt'\end{aligned} \qquad 6.13$$

This equation reflects the change in the sensitive time lag from its steady state value $\bar{\tau}(\bar{z})$ for a particle which begins burning precisely at time t and station z, r, θ after having traveled through the chamber with velocity $V_{\ell}[z'(t'),r'(t'),\theta'(t'),t']$ and having been exposed to pressure perturbations $p'[z'(t'),r'(t'),\theta'(t'),t']$.

Differentiating Eq. 6.13 with respect to t and neglecting higher order terms, there results:

$$\frac{d\tau}{dt}(z,r,\theta,t) = -\frac{n}{\bar{p}}\left\{p'[z,r,\theta,t] - p'[z'(t-\tau),r'(t-\tau),\theta'(t-\tau),t-\tau\right\} \qquad 6.14$$

or equivalently,

$$\frac{d\tau}{dt}(z,r,\theta,t) = -\frac{n}{\bar{p}}\left\{p'[z,r,\theta,t] - P'[\xi(z,r,\theta,t),K(z,r,\theta,t),\chi(z,r,\theta,t), t-\tau(z,r,\theta,t)]\right\} \qquad 6.15$$

It should be emphasized that in writing these equations, we are dealing with a specific particle which enters its sensitive time lag at time $t-\tau(z,r,\theta,t)$ (which coincides with station ξ, K, χ) and which burns precisely at time t (the end of the sensitive time lag) at station

z, r, θ. Thus, in examining Eq. 6.15, it is clear that the pressure perturbation is to be evaluated at an upper limit corresponding to the instant of burning, and at a lower limit representing the beginning of the sensitive time lag. We will now see why the quantity $\frac{d\tau}{dt}$ must be known if we are interested in determining the burning rate perturbation which can occur at a given location in the chamber.

Let $\dot{m}_i$ denote the propellant injection rate, and $\dot{m}_b$ the propellant burning rate, and now consider the fraction of injected propellants burning in unsteady state between stations z and $z+dz$, r and $r+dr$, θ and $\theta+d\theta$. Call this fraction $\delta\dot{m}_b$ and assume that the total time lag $\tau_t(z,r,\theta,t)$ which this fraction experienced in reaching z, r, θ is the same for all particles or elements within the fraction $\delta\dot{m}_b(z,r,\theta,t)$. Now this fraction was injected at time $t-\tau_t(z,r,\theta,t)$ as a fraction $\delta\dot{m}_i$ of the injection mass flow rate $\dot{m}_i$. Then since the fraction which burns in time dt was injected in time $d(t-\tau_t)$, it is clear that the conservation of mass yields:

$$\delta\dot{m}_i\left(t-\tau_t(z,r,\theta,t)\right)d(t-\tau_t) = \delta\dot{m}_b(z,r,\theta,t)\,dt \qquad 6.16$$

and hence we must have:

$$\delta\dot{m}_i\left(t-\tau_t(z,r,\theta,t)\right)\left\{1-\frac{d\tau}{dt}(z,r,\theta,t)\right\} = \delta\dot{m}_b(z,r,\theta,t) \qquad 6.17$$

since although $\bar{\tau}_i$ varies with location, it does not vary with time. In the steady state, $\frac{d\tau}{dt}$ vanishes and we obtain simply:

$$\overline{\delta\dot{m}_i} = \overline{\delta\dot{m}_b} = const. \qquad 6.18$$

In an investigation of intrinsic instability, the injection rate is constant and then we also have:

$$\delta\dot{m}_i = \overline{\delta\dot{m}_i} = \overline{\delta\dot{m}_b} \qquad 6.19$$

If we introduce ϕ as the instantaneous rate per unit volume at which gas is produced at any point in the chamber, then

$$\delta \dot{m}_b(z,r,\theta,t) = r\,\phi(z,r,\theta,t)\; d\theta\, dr\, dz \qquad 6.20$$

by definition of the burning rate of the fraction considered. In the steady state, this same fraction would have burned between stations $\bar{z}$ and $\bar{z}+d\bar{z}$, $\bar{r}$ and $\bar{r}+d\bar{r}$, $\bar{\theta}$ and $\bar{\theta}+d\bar{\theta}$, and again by definition, its burning rate is:

$$\overline{\delta \dot{m}_b}\,(\bar{z},\bar{r},\bar{\theta}) = \bar{r}\,\bar{\phi}\,(\bar{z},\bar{r},\bar{\theta})\, d\bar{\theta}\, d\bar{r}\, d\bar{z} \qquad 6.21$$

and now combining Eqs. 6.17, 6.19, 6.20 and 6.21 we find:

$$r\phi(z,r,\theta,t)\,d\theta\,dr\,dz = \bar{r}\,\bar{\phi}(\bar{z},\bar{r},\bar{\theta})\,d\bar{\theta}\,d\bar{r}\,d\bar{z}\left\{1 - \frac{d\tau}{dt}(z,r,\theta,t)\right\} \qquad 6.22$$

In words, this equation says that at a given instant of time t , the fraction of injected propellants which burns at a particular location z , r , θ is a function of that fraction of injected propellants which burns in steady state at location $\bar{z}$, $\bar{r}$, $\bar{\theta}$, and of the variation in the time lag which occurs as a result of the pressure fluctuations during the coordinating process. (see Fig. 6.2)

Separation of the variables enables us to write:

$$\phi(z,r,\theta,t) = \bar{\phi}(z) + \frac{dg}{dz}(z)\,\Psi(r)\,\Phi(\theta)\,e^{st} \qquad 6.23$$

where $\quad \bar{\phi} = \dfrac{d}{dz}(\bar{V}_z)$

and thus Eq. 6.22 now becomes:

$$r\,\frac{d\bar{V}_z}{dz}(z)\,d\theta\,dr\,dz + \frac{dg}{dz}(z)\,r\,\Psi(r)\,\Phi(\theta)\,e^{st}\,d\theta\,dr\,dz = \qquad 6.24$$

$$\bar{r}\,\frac{d\bar{V}_z}{d\bar{z}}(\bar{z})\,d\bar{\theta}\,d\bar{r}\,d\bar{z} - \bar{r}\,\frac{d\bar{V}_z}{d\bar{z}}(\bar{z})\,\frac{d\tau}{dt}(z,r,\theta,t)\,d\bar{\theta}\,d\bar{r}\,d\bar{z}$$

Let us now relate the area element $\bar{r}d\bar{\theta}d\bar{r}$ which is the steady state area element, to the area element $rd\theta dr$ pertinent to unsteady operation. Introducing the first two Eqs. 6.6, we may write:

$$\int_{r_0}^{r} dr' = \int_0^z \frac{V_{\ell r}'[z', r'(z'), \theta'(z'), t'(z')]}{V_{\ell z}[z', r'(z'), \theta'(z'), t'(z')]} dz' \qquad 6.25$$

where 0, r_0, θ_0 are the coordinates of the point of injection. See Fig. 6.1. Noting that $r_0 \equiv \bar{r}$, we may neglect higher order terms and write:

$$r - \bar{r} = \int_0^z \frac{V_{\ell r}'[z', r'(z'), \theta'(z'), t'(z')]}{\bar{V}_{\ell z}(z')} dz' \qquad 6.26$$

Replacing $V_{\ell r}'$ by the separation variables given in Eq. 5.1 we have:

$$r - \bar{r} = \int_0^z \frac{\eta_r(z') \frac{d\psi}{dr'}[r'(z')]\, \Phi[\theta'(z')]\, e^{st'(z')}}{\bar{V}_{\ell z}(z')} dz' \qquad 6.27$$

Now $t'(z')$ is actually given by Eq. 6.9, but for the purposes of evaluating a perturbation, we may take:

$$t'(z') = t - \int_{z'}^{z} \frac{dz''}{\bar{V}_{\ell z}(z'')} \qquad 6.28$$

and hence Eq. 6.27 becomes:

$$r - \bar{r} = \frac{d\psi}{dr}(r)\, \Phi(\theta)\, e^{st} \int_0^z \frac{\eta_r(z')\, e^{-s\int_{z'}^{z} \frac{dz''}{\bar{V}_{\ell z}(z'')}}}{\bar{V}_{\ell z}(z')} dz' \qquad 6.29$$

on neglecting second order terms. On integrating Eq. 6.29 by parts, we obtain, correct to terms of O(M),

$$r - \bar{r} = \frac{d\psi}{dr}(r)\, \Phi(\theta)\, e^{st} \varphi_0 \frac{k}{s^2}\left[\frac{\nu_r(z)}{\varphi_0} - \frac{\nu_{r_0}}{\varphi_0} e^{-s\int_0^z \frac{dz'}{\bar{V}_{\ell z}(z')}}\right] \qquad 6.30$$

Noting that $\nu_0 = \nu_r$ and utilizing Eqs. 6.6, we may likewise show that:

$$r(\theta-\bar{\theta}) = \frac{\Psi(r)}{r}\frac{d\Phi(\theta)}{d\theta}e^{st}\varphi_0\frac{k}{S^2}\left[\frac{\nu_r(z)}{\varphi_0} - \frac{\nu_{r_0}}{\varphi_0}e^{-s\int_0^z \frac{dz'}{\bar{V}_{\ell z}(z')}}\right] \qquad 6.31$$

Differentiating Eqs. 6.30 and 6.31 we have:

$$d\bar{r} = dr\left[1-\frac{k}{S^2}\varphi_0\left\{\frac{\nu_r(z)}{\varphi_0}-\frac{\nu_{r_0}}{\varphi_0}e^{-s\int_0^z \frac{dz'}{\bar{V}_{\ell z}(z')}}\right\}\frac{d^2\Psi}{dr^2}(r)\,\Phi(\theta)e^{st}\right] \qquad 6.32$$

$$d\bar{\theta} = d\theta\left[1-\frac{k}{S^2}\varphi_0\left\{\frac{\nu_r(z)}{\varphi_0}-\frac{\nu_{r_0}}{\varphi_0}e^{-s\int_0^z \frac{dz'}{\bar{V}_{\ell z}(z')}}\right\}\frac{\Psi(r)}{r^2}\frac{d^2\Phi}{d\theta^2}e^{st}\right] \qquad 6.33$$

and rearranging Eq. 6.30 gives:

$$\bar{r} = r\left[1-\frac{k}{S^2}\varphi_0\left\{\frac{\nu_r(z)}{\varphi_0}-\frac{\nu_{r_0}}{\varphi_0}e^{-s\int_0^z \frac{dz'}{\bar{V}_{\ell z}(z')}}\right\}\frac{d\Psi(r)}{r\,dr}\Phi(\theta)e^{st}\right] \qquad 6.34$$

Taking the product of Eqs. 6.32, 6.33 and 6.34 and neglecting higher order terms, we find:

$$\bar{r}d\bar{\theta}d\bar{r} = r\,d\theta\,dr\left[1-\frac{k}{S^2}\varphi_0\left\{\frac{\nu_r(z)}{\varphi_0}-\frac{\nu_{r_0}(z)}{\varphi_0}e^{-s\int_0^z \frac{dz'}{\bar{V}_{\ell z}(z')}}\right\}\Psi(r)\Phi(\theta)e^{st}\right. \qquad 6.35$$

$$\left.\cdot\left[\frac{d^2\Psi(r)}{\Psi(r)dr^2}+\frac{d\Psi(r)}{r\Psi(r)dr}+\frac{d^2\Phi(\theta)}{r^2\Phi(\theta)d\theta^2}\right]\right]$$

and then introducing the right hand side of Eq. 5.18, we finally obtain:

$$\bar{r}d\bar{\theta}d\bar{r} = r\,d\theta\,dr\left[1+\frac{k}{S^2}s_{nh}^2\varphi_0\left\{\frac{\nu_r(z)}{\varphi_0}-\frac{\nu_{r_0}}{\varphi_0}e^{-s\int_0^z \frac{dz'}{\bar{V}_{\ell z}(z')}}\right\}\Psi(r)\Phi(\theta)e^{st}\right] \qquad 6.36$$

Now introducing this result back into Eq. 6.24 we find:

$$\frac{d\bar{V}_z}{dz}(z)\,dz + \frac{dg}{dz}(z)\,\Psi(r)\,\Phi(\theta)e^{st}dz = \qquad 6.37$$

$$\frac{d\bar{V}_z}{d\bar{z}}(\bar{z})\,d\bar{z}\left[1+\frac{k}{S^2}s_{nh}^2\varphi_0\left\{\frac{\nu_r(z)}{\varphi_0}-\frac{\nu_{r_0}}{\varphi_0}e^{-s\int_0^z \frac{dz'}{\bar{V}_{\ell z}(z')}}\right\}\Psi(r)\Phi(\theta)e^{st}\right]$$

$$-\frac{d\bar{V}_z}{d\bar{z}}(\bar{z})\,d\bar{z}\left[\frac{d\tau}{dt}(z,r,\theta,t)\right]\left[1+\frac{k}{S^2}s_{nh}^2\varphi_0\left\{\frac{\nu_r(z)}{\varphi_0}-\frac{\nu_{r_0}}{\varphi_0}e^{-s\int_0^z \frac{dz'}{\bar{V}_{\ell z}(z')}}\right\}\Psi(r)\Phi(\theta)e^{st}\right]$$

Expanding, eliminating higher order terms and regrouping,

$$\frac{d\varphi}{dz}(z)\,\Psi(r)\,\Phi(\theta)e^{st}\,dz = -\frac{d\bar{V}}{dz}(z)\,dz + \frac{d\bar{V}}{d\bar{z}}(\bar{z})\,d\bar{z} \qquad 6.38$$

$$+\frac{d\bar{V}}{d\bar{z}}(\bar{z})\,\frac{k}{s^2}\,s^2 r\,\varphi_0\left\{\frac{\nu_r(z)}{\varphi_0} - \frac{\nu_{r_0}}{\varphi_0}\,e^{-s\int_0^z \frac{dz'}{\bar{v}_{\ell z}(z')}}\right\}\Psi(r)\,\Phi(\theta)e^{st}\,dz$$

$$-\frac{d\bar{V}}{d\bar{z}}(\bar{z})\,d\bar{z}\;\frac{d\tau}{dt}(z,r,\theta,t)$$

Turning our attention, we proceed to an examination of $\frac{d\tau}{dt}$ as given by Eq. 6.15. From our separation of the variables, we may write:

$$p'[z,r,\theta,t] = \varphi(z)\,\Psi(r)\,\Phi(\theta)e^{st} \qquad 6.39$$

$$p'[\xi(z,r,\theta,t),K(\;),\chi(\;),t-\tau(\;)] = \varphi[\xi(\;)]\,\Psi[K(\;)]\,\Phi[\chi(\;)]\,e^{s\{t-\tau(\;)\}} \qquad 6.40$$

Expanding each factor into a Taylor series, and neglecting products of perturbations, we obtain:

$$p'[z,r,\theta,t] = \varphi(\bar{z})\,\Psi(r)\,\Phi(\theta)e^{st} + \text{higher order terms} \qquad 6.41$$

$$p'[\xi(\;),K(\;),\chi(\;),t-\tau(\;)] = \varphi[\bar{\xi}(\bar{z})]\,\Psi(r)\,\Phi(\theta)e^{s\{t-\bar{\tau}(\bar{z})\}} + \text{higher order terms} \qquad 6.42$$

And now substituting into Eq. 6.15 we have:

$$\frac{d\tau}{dt}(z,r,\theta,t) = -\frac{n}{\bar{p}}\left\{\varphi(\bar{z}) - \varphi[\bar{\xi}(\bar{z})]e^{-s\bar{\tau}(\bar{z})}\right\}\Psi(r)\,\Phi(\theta)e^{st} \qquad 6.43$$

Therefore Eq. 6.38 becomes:

$$\frac{dq}{d\bar{z}}(z)\,\psi(r)\,\Phi(\theta)e^{st}\,dz = -\frac{d\bar{V}}{dz}(z)\,dz + \frac{d\bar{V}}{d\bar{z}}(\bar{z})\,d\bar{z} \tag{6.44}$$

$$+\frac{d\bar{V}_z}{d\bar{z}}(\bar{z})\,\frac{k}{s^2}\,s^2 nh\,\varphi_0\left\{\frac{\nu_r(\bar{z})}{\varphi_0} - \frac{\nu_{r_0}}{\varphi_0}e^{-s\bar{\tau}_t(\bar{z})}\right\}\psi(r)\,\Phi(\theta)e^{st}\,d\bar{z}$$

$$+\frac{n}{\bar{p}}\,\frac{d\bar{V}_z}{d\bar{z}}(\bar{z})\,\varphi_0\left\{\frac{\varphi(\bar{z})}{\varphi_0} - \frac{\varphi}{\varphi_0}\left[\bar{\xi}(\bar{z})\right]e^{-s\bar{\tau}(\bar{z})}\right\}\psi(r)\,\Phi(\theta)e^{st}\,d\bar{z}$$

Let us now integrate each term from zero to the appropriate upper limit, i.e. z or $\bar{z}$ as the case may be, since there is a one-to-one correspondence between z and $\bar{z}$ over their respective paths of integration when the time is held fixed. We write:

$$\psi(r)\,\Phi(\theta)e^{st}\int_0^{z}\frac{dq}{dz'}(z')\,dz' = -\int_0^{z}\frac{d\bar{V}_z}{dz'}(z')\,dz' + \int_0^{\bar{z}}\frac{d\bar{V}_z}{dz'}(z')\,dz' \tag{6.45}$$

$$+\psi(r)\,\Phi(\theta)e^{st}\int_0^{\bar{z}}\frac{k}{s^2}\,s^2 nh\,\varphi_0\left\{\frac{\nu_r(z')}{\varphi_0} - \frac{\nu_{r_0}}{\varphi_0}e^{-s\bar{\tau}_t(z')}\right\}\frac{d\bar{V}_z}{dz'}(z')\,dz'$$

$$+\psi(r)\,\Phi(\theta)e^{st}\int_0^{\bar{z}}\frac{n}{\bar{p}}\,\varphi_0\left\{\frac{\nu_r(z')}{\varphi_0} - \frac{\varphi}{\varphi_0}\left[\bar{\xi}(z')\right]e^{s\bar{\tau}(z')}\right\}\frac{d\bar{V}_z}{dz'}(z')\,dz'$$

Noting that $\bar{V}_z(0) = 0 = q(0)$ we obtain:

$$\left[\psi(r)\,\Phi(\theta)e^{st}\right]q(z) = -\left(\bar{V}_z(z) - \bar{V}_z(\bar{z})\right) \tag{6.46}$$

$$+\left[\psi(r)\,\Phi(\theta)e^{st}\right]\Bigg[\int_0^{\bar{z}}\frac{k}{s^2}\,s^2 nh\,\varphi_0\left\{\frac{\nu_r(z')}{\varphi_0} - \frac{\nu_{r_0}}{\varphi_0}e^{-s\bar{\tau}_t(z')}\right\}\frac{d\bar{V}_z}{dz'}(z')\,dz'$$

$$+\int_0^{\bar{z}}\frac{n}{\bar{p}}\,\varphi_0\left\{\frac{\varphi(z')}{\varphi_0} - \frac{\varphi}{\varphi_0}\left[\bar{\xi}(z')\right]e^{s\bar{\tau}(z')}\right\}\frac{d\bar{V}_z}{dz'}(z')\,dz'$$

From a Taylor expansion,

$$\bar{V}_z(z) = \bar{V}_z(\bar{z}) + \frac{d\bar{V}_z}{d\bar{z}}(\bar{z})\left\{z - \bar{z}\right\} + \cdots\cdots \tag{6.47}$$

and hence by analogy with the one-dimensional treatment in Ref. 18, pages 112-113, it may be shown that:

$$\bar{V}_z(z) - \bar{V}_z(\bar{z}) = \frac{d\bar{V}}{d\bar{z}}(\bar{z})\,\bar{V}_{\ell z}(z)\left\{\int_0^z \frac{V_{\ell z}'[z', r'(z') \ldots]\,dz'}{\bar{V}_{\ell z}^2(z')} \right. \tag{6.48}$$

$$\left. - n\int_{\bar{\xi}}^{z} \frac{p'[z', r'(z'), \theta'(z'), t'(z')]\,dz'}{\bar{p}\ \bar{V}_{\ell z}(z')}\,dz'\right\}$$

$$= \frac{d\bar{V}_z}{d\bar{z}}(\bar{z})\,\bar{V}_{\ell z}(z)\,\psi(r)\,\Phi(\theta)\,e^{st}\left\{\int_0^z \frac{\eta_z(z')\,e^{s\int_{z'}^{z}\frac{dz''}{\bar{V}_{\ell z}(z'')}}}{\bar{V}_{\ell z}^2(z')}\,dz' \right.$$

$$\left. - n\int_{\bar{\xi}}^{z} \frac{\varphi(z')\,e^{s\int_z^{z'}\frac{dz''}{\bar{V}_{\ell z}(z'')}}}{\bar{p}\ \bar{V}_{\ell z}(z')}\,dz'\right\}$$

so that on introducing the latter into Eq. 6.46 we find that the factor $\psi(r)\Phi(\theta)e^{st}$ may be eliminated. Hence,

$$\frac{g(z)}{\varphi_0} = \int_0^{\bar{z}} \frac{n}{\bar{p}}\left\{\frac{\varphi(z')}{\varphi_0} - \frac{\varphi}{\varphi_0}[\bar{\xi}(z')]\,e^{-s\bar{\tau}(z')}\right\}\frac{d\bar{V}_z}{dz'}(z')\,dz' \tag{6.49}$$

$$+ \frac{d\bar{V}_z}{d\bar{z}}(\bar{z})\,\bar{V}_{\ell z}(z)\left\{n\int_{\bar{\xi}}^{z}\frac{\varphi(z')}{\varphi_0}\frac{e^{s\int_z^{z'}\frac{dz''}{\bar{V}_{\ell z}(z'')}}}{\bar{V}_{\ell z}(z')}\,dz' - \int_0^z \frac{\eta_z(z')\,e^{s\int_z^{z'}\frac{dz''}{\bar{V}_{\ell z}(z'')}}}{\varphi_0\ \bar{V}_{\ell z}^2(z')}\,dz'\right\}$$

$$+ \frac{k}{S^2}\,s^2 nh\int_0^{\bar{z}}\left\{\frac{v_r(z')}{\varphi_0} - \frac{v_{r0}}{\varphi_0}\,e^{-s\bar{\tau}_t(z')}\right\}\frac{d\bar{V}_z}{dz'}(z')\,dz'$$

And now after integrating by parts and analyzing the orders of magnitude of the resulting terms, it may be shown, as in Ref. 18, that upon discarding terms of $O(M^2)$ or higher, the remaining terms are:

$$\frac{g(z)}{\varphi_0} = n\int_0^z \frac{\varphi(z')}{\varphi_0}\left[1 - e^{-s\bar{\tau}(z')}\right]\frac{d\bar{V}_z}{dz'}(z')\,dz' \tag{6.50}$$

$$+ \frac{n}{s}\,\frac{d\bar{V}_z}{dz}(z)\,\bar{V}_{\ell z}(z)\,\frac{\varphi(z)}{\varphi_0}\left[1 - e^{-s\bar{\tau}(z)}\right]$$

$$- \frac{k}{S^2}\,\frac{v_z(z)}{\varphi_0}\,\frac{d\bar{V}_z}{dz}(z)$$

when it is assumed that

$$\frac{\varphi}{\varphi_0},\ \frac{v_z}{\varphi_0},\ \frac{1}{\varphi_0}\frac{d\varphi}{dz},\ \frac{1}{\varphi_0}\frac{dv_z}{dz},\ \bar{\tau}(z) = O(1) \tag{6.51}$$

7. Solution by Iteration

In Section 5, we derived expressions for $\frac{\varphi(z)}{\varphi_0}$, $\frac{\gamma\nu_z(z)}{\varphi_0}$, $\frac{\gamma\nu_r(z)}{\varphi_0}$ and $\frac{\gamma\epsilon(z)}{\varphi_0}$ which are given respectively by Eqs. 5.51, 5.52, 5.53 and 5.65. We must therefore examine the order of magnitude of the following integrals which appear in the aforementioned expressions.

$$\int_0^z \gamma E(z') \begin{matrix}\sinh\\ \cosh\end{matrix} \sqrt{s^2+s_{nh}^2}\,(z-z')\,dz' \qquad 7.1$$

$$\int_0^z \gamma F(z') \begin{matrix}\sinh\\ \cosh\end{matrix} \sqrt{s^2+s_{nh}^2}\,(z-z')\,dz'$$

$$\int_0^z \gamma G(z') \begin{matrix}\sinh\\ \cosh\end{matrix} \sqrt{s^2+s_{nh}^2}\,(z-z')\,dz'$$

$$\int_0^z \gamma U(z') \begin{matrix}\sinh\\ \cosh\end{matrix} \sqrt{s^2+s_{nh}^2}\,(z-z')\,dz'$$

Where E , F , G , and U are defined through Eqs. 5.32, 5.35 and 5.41. Restricting our attention to the case $S = O(1)$, $S_{nh} = O(1)$, an involved term by term analysis shows that for the purposes of evaluating the four integrals in 7.1, γE , γF , γG and γU may be considerably reduced to:

$$\gamma E(z') = (3-\gamma)\,\bar{V}_z(z')\,\frac{\gamma\nu_z(z')}{\varphi_0} \qquad 7.2$$

$$\gamma F(z') = \frac{\gamma Q(z')}{\varphi_0} - (\gamma-1)\,\bar{V}_z(z')\,\frac{\varphi(z')}{\varphi_0} + \frac{k}{s}\,\bar{\rho}_\ell(z')\,\frac{\gamma\nu_z(z')}{\varphi_0}$$

$$\gamma G(z') = \frac{k}{s}\,\bar{\rho}_\ell(z')\,\frac{\gamma\nu_r(z')}{\varphi_0} - 2\,\frac{\bar{V}_z(z')}{s}\,\frac{\gamma\nu_z(z')}{\varphi_0}$$

$$\gamma U(z') = \bar{V}_z(z')\,\frac{\gamma\nu_r(z')}{\varphi_0}$$

where $$Q(z) = n\int_0^z \frac{\varphi(z')}{\varphi_0}\left[1-e^{-s\bar{\tau}(z')}\right]\frac{d\bar{V}_z}{dz'}(z')\,dz'$$

is the first term in Eq. 6.50.

All terms which do not appear explicitly above, yield contributions of $O(M^2)$ or higher after integration, while those terms which are retained

yield contributions of O(M). Furthermore, γY , γU , γM and $\gamma \frac{dU}{dz}$ are also each O(M) and hence we may write:

$$\frac{\varphi(z)}{\varphi_0} = \cosh\sqrt{s^2+s_{nh}^2}\, z + O(M) \qquad 7.3$$

$$\frac{\gamma \nu_z(z)}{\varphi_0} = -\frac{\sqrt{s^2+s_{nh}^2}}{s} \sinh\sqrt{s^2+s_{nh}^2}\, z + O(M)$$

$$\frac{\gamma \nu_r(z)}{\varphi_0} = -\frac{1}{s} \cosh\sqrt{s^2+s_{nh}^2}\, z + O(M)$$

while examination of Eq. 5.65 shows that the entropy term is:

$$\frac{\gamma \epsilon(z)}{\varphi_0} = O(M) \qquad 7.4$$

This suggests that we take as the zeroth approximation, correct to terms of O(1):

$$\left(\frac{\varphi(z)}{\varphi_0}\right)^{(0)} = \cosh\sqrt{s^2+s_{nh}^2}\, z \qquad 7.5$$

$$\left(\frac{\gamma \nu_z(z)}{\varphi_0}\right)^{(0)} = -\frac{\sqrt{s^2+s_{nh}^2}}{s} \sinh\sqrt{s^2+s_{nh}^2}\, z$$

$$\left(\frac{\gamma \nu_r(z)}{\varphi_0}\right)^{(0)} = -\frac{1}{s} \cosh\sqrt{s^2+s_{nh}^2}\, z$$

$$\left(\frac{\gamma \epsilon(z)}{\varphi_0}\right)^{(0)} = 0$$

and these may then be used in the evaluation of those terms and integrals yielding contributions of O(M).

We note that Eqs. 7.5 constitute an exact solution for the special case of zero Mach number, since in that case, there is no combustion and the equations coincide precisely with the acoustic solution. If we replace the exhaust nozzle with a closed end, so that there is no outflow, then $\frac{\gamma \nu_z(z)}{\varphi_0}$ must be zero at $z = z_e = L$. The phenomenon is thus reduced to the classical acoustic oscillation in a cylinder closed at both ends. On setting

$s = i\omega$ and $\sinh\sqrt{s^2 + s^2_{nh}}\, L = 0$ we obtain the eigenvalues

$$\omega = \sqrt{\frac{m^2\pi^2}{L^2} + S^2_{nh}} \qquad (m = 0, 1, 2, 3 \cdots) \qquad 7.6$$

which are characteristic of modes of acoustic oscillations in cylindrical chambers. Since we have taken $s = i\omega$, those oscillations must be neutral with well-defined frequencies given by Eq. 7.6.

If combustion takes place, and the Mach number is then small but finite, two modifications occur. First, terms of O(M) must now be considered in evaluating the perturbations, and further, the boundary condition at $z = z_e$ for neutral oscillations is no longer given by $\frac{\gamma \nu_{ze}}{\varphi_o} = 0$, but rather,

$$\frac{\gamma \nu_{ze}}{\varphi_o} + \mathcal{A}\frac{\varphi_e}{\varphi_o} + \mathcal{B}\, S_{nh}\frac{\gamma \nu_{re}}{\varphi_o} + \mathcal{C}\frac{\gamma \epsilon_e}{\varphi_o} = 0 \qquad 7.7$$

This relation was derived by Crocco in Ref. 14, and is the extension to three dimensional flows of the general solution obtained by him in Ref. 5. The coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are complex functions of the frequency, mode (S_{nh}), and the nozzle geometry, and the perturbations at the nozzle entrance must bear the amplitude and phase relationship given by Eq. 7.7 if neutral oscillations are to be maintained. We stress that, in our case, we could not legitimately take $\frac{\gamma \nu_{ze}}{\varphi_o} = 0$ (closed end) as a boundary condition even if terms of O(M) were neglected in Eq. 7.3. Hence, since we have a new boundary condition at $z = z_e$ it is clear that the values of ω for neutral oscillations must now be different from the acoustic solution, Eq. 7.6.

Leaving ω for the moment as the unknown eigenvalue to be determined later, we will now set down the expressions for the perturbations

which will be used in conjunction with Eq. 7.7. First we note that:

$$
\begin{aligned}
\gamma M(z_e) &= 0 \qquad\qquad 7.8\\
\gamma W(z_e) &= 2\bar{V}_{ze}\,\frac{\gamma \nu_{ze}}{\varphi_0}\\
\gamma Y(z_e) &= \gamma n\int_0^{z_e}\left(1-e^{-s\bar{\tau}(z')}\right)\frac{\varphi(z')}{\varphi_0}\,\frac{d\bar{V}_z}{dz'}(z')\,dz' - \gamma\bar{V}_{ze}\,\frac{\varphi_e}{\varphi_0}\\
\frac{s^2nh}{s}\,\gamma U(z_e) &= \frac{s^2nh}{s}\,\bar{V}_{ze}\,\frac{\gamma\nu_{re}}{\varphi_0}\\
\frac{\gamma}{s}\,\frac{dU}{dz}(z_e) &= \frac{V_{ze}}{s}\,\frac{d}{dz}\left(\frac{\gamma\nu_r}{\varphi_0}\right)_{ze}
\end{aligned}
$$

where these results are obtained as a consequence of Eq. 5.6 and the fact that combustion is complete at $z = z_e$ so that

$$
\frac{d\bar{V}_z}{dz}(z_e) = \bar{\rho}_\ell(z_e) = \frac{d\bar{\rho}_\ell}{dz}(z_e) = 0
$$

Letting $s = i\omega$ (Investigation of neutral instability), introducing Eqs. 7.5, evaluating the integrals at an upper limit $z = z_e$, and making use of Eqs. 7.8, Eqs. 5.51, 5.52, 5.53 and 5.65 become:

$$
\begin{aligned}
\left(\frac{\varphi_e}{\varphi_0}\right)^{(1)} &= \cosh\sqrt{\quad}\,z_e + \frac{\omega^2}{\sqrt{\quad}}\int_0^{z_e}\gamma E(z')\sinh\sqrt{\quad}(z-z')\,dz' \qquad 7.9\\
&\quad -\frac{2\bar{V}_{ze}\sqrt{\quad}\,i}{\omega}\sinh\sqrt{\quad}\,z_e - i\omega\int_0^{z_e}\gamma F(z')\cosh\sqrt{\quad}(z-z')\,dz'\\
&\quad +\frac{s^2nh}{\sqrt{\quad}}\int_0^{z_e}\gamma G(z')\sinh\sqrt{\quad}(z-z')\,dz'\\
&\quad + 2s^2nh\int_0^{z_e}\gamma U(z')\cosh\sqrt{\quad}(z-z')\,dz'
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\gamma\nu_{ze}}{\varphi_0}\right)^{(1)} &= \frac{i\sqrt{\quad}}{\omega}\sinh\sqrt{\quad}\,z_e + \gamma n\int_0^{z_e}\left(1-e^{-i\omega\bar{\tau}(z')}\right)\cosh\sqrt{\quad}(z-z')\,dz' \qquad 7.10\\
&\quad -\gamma\bar{V}_{ze}\cosh\sqrt{\quad}\,z_e - \frac{s^2nh}{\omega}\,\bar{V}_{ze}\cosh\sqrt{\quad}\,z_e\\
&\quad + i\omega\int_0^{z_e}\gamma E(z')\cosh\sqrt{\quad}(z-z')\,dz' + \sqrt{\quad}\int_0^{z_e}\gamma F(z')\sinh\sqrt{\quad}(z-z')\,dz'\\
&\quad - s^2nh\int_0^{z_e}\gamma G(z')\cosh\sqrt{\quad}(z-z')\,dz'\\
&\quad + 2\sqrt{\quad}\left(\frac{s^2nh}{\omega}\right)i\int_0^{z_e}\gamma U(z')\sinh\sqrt{\quad}(z-z')\,dz'
\end{aligned}
$$

$$\left(\frac{\gamma \nu_{re}}{\varphi_0}\right)^{(1)} = \frac{i}{w}\cosh\sqrt{\quad}\, z_e + \frac{\sqrt{\quad}}{w^2}\, \bar{V}_{ze} \sinh\sqrt{\quad}\, z_e \tag{7.11}$$

$$+ \frac{iw}{\sqrt{\quad}} \int_0^{z_e} \gamma E(z') \sinh\sqrt{\quad}\,(z-z')\,dz'$$

$$+ \int_0^{z_e} \gamma F(z') \cosh\sqrt{\quad}\,(z-z')\,dz'$$

$$+ \frac{s^2_{nh}\, i}{w\sqrt{\quad}} \int_0^{z_e} \gamma G(z') \sinh\sqrt{\quad}\,(z-z')\,dz'$$

$$+ 2\,\frac{s^2_{nh}}{w}\, i \int_0^{z_e} \gamma U(z') \cosh\sqrt{\quad}\,(z-z')\,dz'$$

$$\left(\frac{\gamma \epsilon_e}{\varphi_0}\right)^{(1)} = -(\gamma-1)\left\{ \bar{V}_{ze}\, \frac{\sqrt{\quad}\, i}{w} \sinh\sqrt{\quad}\, z_e \right. \tag{7.12}$$

$$+ \frac{1}{\bar{V}_{ze}} \int_0^{z_e} e^{\,iw \int_{z_e}^{z'} \frac{dz''}{\bar{V}_z(z'')}} \cosh\sqrt{\quad}\, z'\, \frac{d\bar{V}_z}{dz'}(z')\, dz'$$

$$\left. + \frac{\sqrt{\quad}}{\bar{V}_{ze}} \int_0^{z_e} e^{\,iw \int_{z_e}^{z'} \frac{dz''}{\bar{V}_z(z'')}}\, \bar{V}_z(z') \sinh\sqrt{\quad}\, z'\, dz' \right\}$$

where as a consequence of Eq. 7.5,

$$\gamma E(z') = \frac{(3-\gamma)\sqrt{\quad}\, i}{w}\, \bar{V}_z(z') \sinh\sqrt{\quad}\, z' \tag{7.13}$$

$$\gamma F(z') = \gamma n \int_0^{z'} \left(1 - e^{-iw\bar{\tau}(z')}\right) \cosh\sqrt{\quad}\, z''\, \frac{d\bar{V}_z}{dz''}(z'')\,dz''$$

$$-(\gamma-1)\,\bar{V}_z(z') \cosh\sqrt{\quad}\, z' + \frac{k}{w^2}\,\bar{\rho}_e(z')\sqrt{\quad}\,\sinh\sqrt{\quad}\, z'$$

$$\gamma G(z') = \frac{k}{w^2}\,\bar{\rho}_e(z') \cosh\sqrt{\quad}\, z' - \frac{2\sqrt{\quad}}{w^2}\,\bar{V}_z(z') \sinh\sqrt{\quad}\, z'$$

$$\gamma U(z') = \frac{i}{w}\,\bar{V}_z(z') \cosh\sqrt{\quad}\, z'$$

and $\sqrt{\quad} \equiv \sqrt{s^2 + s^2_{nh}} = \sqrt{-w^2 + s^2_{nh}}$

We observe that when $w > S_{nh}$ then, $\sqrt{\quad} = i\sqrt{w^2 - s^2_{nh}}$

and

$$\cosh\sqrt{\quad} = \cos\sqrt{w^2 - s^2_{nh}}$$

$$\sinh\sqrt{\quad} = i\sin\sqrt{w^2 - s^2_{nh}}$$

The result of this first iteration is correct up to terms of O(M) provided that S and S_{nh} are O(1). In principle, one could proceed with additional iterations, however, the net result would be to introduce terms of $O(M^2)$, a refinement which is not required if the Mach number is sufficiently small, in the neighborhood of 0.10, say.

8. Solution for the Eigenvalues

The stability problem can be stated as follows: for a given chamber geometry, distribution of combustion, and exhaust nozzle, will an arbitrary perturbation of the steady state conditions be amplified or damped?

But, the steady state distribution of combustion is represented by $\bar{V}_z(z)$ since $\bar{\phi} = \frac{d\bar{V}_z}{dz}$, while the unsteady effects of combustion are represented by the distribution of the sensitive and total time lags $\tau(z)$ and $\tau_t(z)$, and by the interaction index n. Hence, stated concisely, for a given $\bar{V}_z$, $\bar{\tau}$, $\bar{\tau}_t$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and n, will arbitrary perturbations be amplified or damped?

Mathematically, the answer to this question is given by analyzing the real part Λ of $s = \Lambda + i\Omega$. In practice, we need only determine the neutral condition under which Λ changes its sign ($\Lambda = 0$), the stability boundary. If $\bar{V}_z$, $\bar{\tau}_t$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are fixed, neutral conditions will be possible only when a certain relation involving $\bar{\tau}$ and the interaction index n is satisfied, and further, these neutral oscillations will take place with a well determined frequency.

Equation 7.7 represents the functional relationship between the three quantities, and since it is complex, it corresponds to two real equations. If for simplicity we assume that the sensitive time lag is the same for all elements, then we have two relations between the three

quantities $\bar{\delta}$, n and ω, where $\bar{\delta}$ is the critical value of the sensitive time lag. This means that for a given value of n, Eq. 7.7 will determine the values of the time lag $\bar{\delta}$ and the frequency ω for which neutral oscillations can be obtained. In other words, Eq. 7.7 represents the characterisitic equation for the set of eigenvalue $\bar{\delta}$ and ω. The most convenient procedure, however, is to prescribe the value of ω and solve for the eigenvalues n and $\bar{\delta}$ compatible with neutral oscillations for that value of ω.

Now introducing Eqs. 7.9, 7.10, 7.11 and 7.12 into Eq. 7.7 and introducing the notation:

$$E_c = \int_0^{z_e} \gamma E(z') \cosh\sqrt{}\,(z-z')\,dz' \qquad 8.1$$

$$E_s = \int_0^{z_e} \gamma E(z') \sinh\sqrt{}\,(z-z')\,dz'$$

$$G_c = \int_0^{z_e} \gamma G(z') \cosh\sqrt{}\,(z-z')\,dz'$$

$$G_s = \int_0^{z_e} \gamma G(z') \sinh\sqrt{}\,(z-z')\,dz'$$

$$U_c = \int_0^{z_e} \gamma U(z') \cosh\sqrt{}\,(z-z')\,dz'$$

$$U_s = \int_0^{z_e} \gamma U(z') \sinh\sqrt{}\,(z-z')\,dz'$$

$$V_c = \int_0^{z_e} \cosh\sqrt{}\,(z-z')\cdot\bar{V}_z(z')\cdot\cosh\sqrt{}\,z'\,dz'$$

$$V_s = \int_0^{z_e} \sinh\sqrt{}\,(z-z')\cdot\bar{V}_z(z')\cosh\sqrt{}\,z'\,dz'$$

$$R_c = \int_0^{z_e} \cosh\sqrt{}\,(z-z')\,\bar{\rho}_\ell(z')\sinh\sqrt{}\,z'\,dz'$$

$$R_s = \int_0^{z_e} \sinh\sqrt{}\,(z-z')\,\bar{\rho}_\ell(z')\sinh\sqrt{}\,z'\,dz'$$

we obtain:

$$h_2 n\left(1-e^{-i\omega\bar{\delta}}\right) + h_3 = 0 \qquad 8.2$$

where h_2 and h_3 are complex functions given by:

$$h_2 = \gamma\int_0^{z_e}\cosh\sqrt{\Gamma}\, z'\,\frac{d\bar{V}_z}{dz'}(z')\,dz' \qquad 8.3$$

$$+\gamma\sqrt{\Gamma}\int_0^{z_e}\sinh\sqrt{\Gamma}\,(z-z')\left\{\int_0^{z'}\cosh\sqrt{\Gamma}\,z''\,\frac{d\bar{V}_z(z'')}{dz''}\,dz''\right\}dz'$$

$$+(\gamma\mathcal{B}\,S\,nh - i\omega\gamma\mathcal{A})\int_0^{z_e}\cosh\sqrt{\Gamma}\,(z-z')\left\{\int_0^{z'}\cosh\sqrt{\Gamma}\,z''\,\frac{d\bar{V}_z}{dz''}(z'')\,dz''\right\}dz'$$

$$h_3 = \left[\frac{i\sqrt{\Gamma}}{\omega}\left(1-2\bar{V}_{ze}\mathcal{A}\right)+\frac{\mathcal{B}\,S\,nh\sqrt{\Gamma}}{\omega^2}\bar{V}_{ze}-(\gamma-1)\frac{C\bar{V}_{ze}\sqrt{\Gamma}\,i}{\omega}\right]\sinh\sqrt{\Gamma}\,z_e \qquad 8.4$$

$$+\left[\mathcal{A}-\left(\gamma+\frac{S^2nh}{\omega^2}\right)\bar{V}_{ze}+\frac{\mathcal{B}\,S\,nh\,i}{\omega}\right]\cosh\sqrt{\Gamma}\,z_e$$

$$+\,i\omega E_c+\left[\frac{\omega^2}{\sqrt{\Gamma}}\mathcal{A}+\frac{i\omega}{\sqrt{\Gamma}}\mathcal{B}\,S\,nh\right]E_s$$

$$-\sqrt{\Gamma}(\gamma-1)V_s+\left[i\omega(\gamma-1)\mathcal{A}-(\gamma-1)\mathcal{B}\,S\,nh\right]V_c$$

$$+\frac{S^2nh-\omega^2}{\omega^2}k\,R_s+\left[\sqrt{\Gamma}\frac{k}{\omega^2}\mathcal{B}\,S\,nh-i\omega\sqrt{\Gamma}\frac{k}{\omega^2}\mathcal{A}\right]R_c$$

$$-\,S^2nh\,G_c+\left[\frac{S^2nh}{\sqrt{\Gamma}}\mathcal{A}+\frac{S^2nh}{\omega\sqrt{\Gamma}}i\,\mathcal{B}\,S\,nh\right]G_s$$

$$+\,2\sqrt{\Gamma}\,\frac{S^2nh\,i}{\omega}U_s+\left[2S^2nh\,\mathcal{A}+\frac{2S^2nh\,i}{\omega}\mathcal{B}\,S\,nh\right]U_c$$

$$-(\gamma-1)\,C\left\{\frac{1}{\bar{V}_{ze}}\int_0^{z_e}e^{i\omega\int_{z_e}^{z'}\frac{dz''}{\bar{V}_z(z'')}}\cosh\sqrt{\Gamma}\,z'\,\frac{d\bar{V}_z}{dz'}(z')\,dz'\right.$$

$$\left.+\frac{\sqrt{\Gamma}}{\bar{V}_{ze}}\int_0^{z_e}e^{i\omega\int_{z_e}^{z'}\frac{dz''}{\bar{V}_z(z'')}}\bar{V}_z(z')\sinh\sqrt{\Gamma}\,z'\,dz'\right\}$$

Now, we have demonstrated that h_2 and h_3 are uniquely determined once we have specifica the rocket chamber geometry, the distribution of combustion and the exhaust nozzle geometry. Hence Eq. 8.2 is the final form of the characteristic equation and may be used in the study of the neutral stability of a given rocket system.

Letting,

$$a_1 = h_{2Im} h_{3Re} - h_{2Re} h_{3Im} \tag{8.5}$$

$$a_2 = -h_{2Im} h_{3Im} - h_{2Re} h_{3Re}$$

we may eliminate η from Eq. 8.2 to obtain:

$$a_1 \cos \omega \bar{\delta} + a_2 \sin \omega \bar{\delta} = a_1 \tag{8.6}$$

The trivial solution $\omega \bar{\delta} = 0, 2\pi, 4\pi \ldots$ is discarded, and we find that the solution is given by the simultaneous set:

$$\sin \omega \bar{\delta} = \frac{2 a_1 a_2}{a_1^2 + a_2^2} = \sin A \tag{8.7}$$

$$\cos \omega \bar{\delta} = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2} = \cos A$$

and hence,

$$\bar{\delta} = \frac{1}{\omega} \left[A + 2K\pi \right] \qquad (K = 0, 1, 2 \ldots) \tag{8.8}$$

Having determined one of the eigenvalues, we may determine the other by substituting into either of the equivalent forms:

$$\eta = \frac{-h_{3Re}}{h_{2Re}(1 - \cos \omega \bar{\delta}) - h_{2Im} \sin \omega \bar{\delta}} \tag{8.9}$$

$$\eta = \frac{-h_{3Im}}{h_{2Im}(1 - \cos \omega \bar{\delta}) + h_{2Re} \sin \omega \bar{\delta}}$$

And thus we have solved for the two eigenvalues n and $\tilde{\delta}$.

We note that when the rates of the physico-chemical processes depend on the mixture ratio, another interaction index exists and a more complicated form of the characteristic equation is obtained. In the treatment of the entropy wave instability analysis, it will be seen that Eq. 8.2 is a special case of a more general relationship involving both pressure correlated and mixture correlated effects.

III. ENTROPY WAVE INSTABILITY

9. The Entropy Wave Equations

In the previous sections, we were concerned with an investigation of transverse modes of combustion instability, in which the coordinating mechanism depended primarily on the presence of transverse pressure waves. Although entropy terms were considered in that analysis, which certainly influences the resulting magnitude of the interaction index n and the critical value of the sensitive time lag $\bar{\delta}$, the instability itself could not be attributed to the presence of entropy waves, because these waves must travel down the chamber with the speed of the mean gas motion, and hence a consideration of the total period involved indicates that they can yield instability only at intermediate values of the chamber frequency, i.e. $O(M) < \omega < O(1)$. Since the derived frequencies were of order unity and, furthermore could be correlated with an acoustic mode, this would imply that in our previous analysis, the responsible agent must be of an acoustic nature.

As we pointed out in Section 2, in which we discussed the status of the theory, entropy waves may be formed in either of two ways, and if both mechanisms exist simultaneously, they will reinforce each other.

For example, if we are dealing with a monopropellant motor, entropy waves can be produced directly by chamber pressure oscillations, since neighboring propellant elements will combust to a final temperature determined primarily by the steady state mixture ratio, however, because the pressure is different, the two sources of gas will each have a different final entropy. At any instant of time, the distribution of excess pressure in the chamber is wave shaped, and hence the instantaneous distribution of

entropy production in the chamber likewise forms a wave, and this wave travels down the chamber and reflects pressure waves at the nozzle exit. Thus, a closed loop now exists and yields a mechanism for combustion instability, since the pressure waves will proceed to generate new entropy waves.

If we are dealing with a bipropellant rocket motor, mixture ratio variations may be the chief cause of entropy wave instability, since at these frequencies, the injection system will respond to chamber pressure oscillations and can produce off-ratio mixtures because the oxidizer and propellant lines can respond differently. In this case, entropy waves will be produced because the combustion temperature will be different for neighboring sources of gas. These entropy waves will travel down the chamber with the mean gas velocity and will reflect pressure waves at the exhaust nozzle which again form a closed loop.

The total period will depend on the sum of the entropy wave travel time downstream and the pressure wave travel time upstream plus whatever time is involved for the propellant element to reach the combustion front. (See Fig. 9.1)

In the following sections, we propose to investigate combustion instability in the intermediate frequency range as it is caused by entropy waves. We shall therefore consider the effect of mixture ratio variations as might occur in a liquid bipropellant rocket motor. For completeness, we will also consider the possibility of obtaining entropy wave instability in the absence of injector response. The latter may be termed intrinsic entropy wave instability.

In Section 3, we derived a set of partial differential equations which are applicable to a generalized study of combustion instability in liquid propellant rockets. These are repeated below for convenience. We

have as follows:

Conservation of Mass

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \underline{V}^*) = \phi^* = -\frac{\partial \rho_\ell^*}{\partial t^*} - \nabla^* \cdot (\rho_\ell^* \underline{V}_\ell^*) \tag{9.1}$$

Conservation of Momentum

$$\frac{\partial}{\partial t^*}(\rho^* \underline{V}^* + \rho_\ell^* \underline{V}_\ell^*) + \rho^* \underline{V}^* (\nabla^* \cdot \underline{V}^*) + (\underline{V}^* \cdot \nabla^*) \rho^* \underline{V}^* \tag{9.2}$$

$$+ \rho_\ell^* \underline{V}_\ell^* (\nabla^* \cdot \underline{V}_\ell^*) + (\underline{V}_\ell^* \cdot \nabla^*) \rho_\ell^* \underline{V}_\ell^* = -\nabla^* p^*$$

Conservation of Energy

$$\frac{\partial}{\partial t^*}(\rho^* h_s^* + \rho_\ell^* h_{\ell s}^*) + \nabla^* \cdot (\rho^* h_s^* \underline{V}^* + \rho_\ell^* h_{\ell s}^* \underline{V}_\ell^*) = \frac{\partial p^*}{\partial t^*} \tag{9.3}$$

Equation of State

$$p^* = \rho^* \mathcal{R}^* T^* \tag{9.4}$$

Droplet Dynamics

$$\frac{d\underline{V}_\ell^*}{dt^*} = k^*(\underline{V}^* - \underline{V}_\ell^*) \tag{9.5}$$

Droplet Stagnation Enthalpy

$$\frac{dh_{\ell s}^*}{dt^*} = 0 \tag{9.6}$$

As we have already stated, entropy wave instability is essentially a longitudinal phenomenon and hence the following treatment will be one-dimensional. We will now let our reference length be the chamber length and we take:

$$\nabla = L^* \nabla^* , \quad t = \frac{\bar{c}_0^* t^*}{L^*} , \quad p = \frac{p^*}{\bar{p}_0^*} , \quad T = \frac{T^*}{\bar{T}_0^*} \tag{9.7}$$

$$\rho = \frac{\rho^*}{\bar{\rho}_0^*} , \quad \rho_\ell = \frac{\rho_\ell^*}{\bar{\rho}_0^*} , \quad V = \frac{V^*}{\bar{c}_0^*} , \quad V_\ell = \frac{V_\ell^*}{\bar{c}_0^*}$$

$$\phi = \frac{L^* \phi^*}{\bar{\rho}_0^* \bar{c}_0^*} , \quad h = \frac{(\gamma - 1) h^*}{\gamma \mathcal{R}^* \bar{T}_0^*} , \quad k = \frac{k^* L^*}{\bar{c}_0^*}$$

The non-dimensional equations then become:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho V) = \phi = -\frac{\partial \rho_\ell}{\partial t} - \frac{\partial}{\partial z}(\rho_\ell V_\ell) \qquad 9.1a$$

$$\frac{\partial}{\partial t}(\rho V + \rho_\ell V_\ell) + \frac{\partial}{\partial z}(\rho V^2 + \rho_\ell V_\ell^2) = -\frac{1}{\gamma}\frac{\partial p}{\partial z} \qquad 9.2a$$

$$\rho\left(\frac{\partial h_s}{\partial t} + V\frac{\partial h_s}{\partial z}\right) = \frac{\gamma-1}{\gamma}\frac{\partial p}{\partial t} - \phi(h_s - h_{\ell s}) \qquad 9.3a$$

$$p = \rho T \qquad 9.4a$$

$$\frac{dV_\ell}{dt} = \frac{\partial V_\ell}{\partial t} + V_\ell\frac{\partial V_\ell}{\partial z} = k(V - V_\ell) \qquad 9.5a$$

$$\frac{dh_{\ell s}}{dt} = 0 \quad ; \quad h_{\ell s} = h_{\ell s_0} = h_\ell + \frac{\gamma-1}{2}V_\ell^2 \qquad 9.6a$$

And now we proceed as before by introducing small perturbations. That is, each of the dependent variables will again be represented by the sum of a steady state space-variable and a time-dependent perturbation so small that terms higher than those linear in the perturbations may be neglected. A superposed bar denotes steady state and a prime denotes a perturbation. The steady state and perturbed equations follow directly.

Mass

$$\frac{d}{dz}(\bar\rho\bar V) = \bar\phi = -\frac{d}{dz}(\bar\rho_\ell \bar V_\ell) \qquad 9.1b$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial z}(\bar\rho V' + \rho'\bar V) = \phi' = -\frac{\partial \rho_\ell'}{\partial t} - \frac{\partial}{\partial z}(\bar\rho_\ell V_\ell' + \rho_\ell'\bar V_\ell) \qquad 9.1c$$

Momentum

$$\frac{d}{dz}\left(\bar{\rho}\bar{V}^2 + \bar{\rho}_\ell \bar{V}_\ell^2\right) = -\frac{1}{\gamma}\frac{d\bar{p}}{dz} \qquad 9.2b$$

$$\bar{\rho}\frac{\partial V'}{\partial t} + \bar{V}\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial z}\left(2\bar{\rho}\bar{V}V' + \rho'\bar{V}^2\right) = -\frac{1}{\gamma}\frac{\partial p'}{\partial z}$$
$$-\bar{\rho}_\ell\frac{\partial V_\ell'}{\partial t} - \bar{V}_\ell\frac{\partial \rho_\ell'}{\partial t} - \frac{\partial}{\partial z}\left(2\bar{\rho}_\ell\bar{V}_\ell V_\ell' + \rho_\ell'\bar{V}_\ell^2\right) \qquad 9.2c$$

Energy

$$\bar{\rho}\bar{V}\frac{d\bar{h}_s}{dz} = \left(\bar{h}_{\ell s} - \bar{h}_s\right)\bar{\phi} = 0 \qquad 9.3b$$

$$\bar{\rho}\left(\frac{\partial h_s'}{\partial t} + \bar{V}\frac{\partial h_s'}{\partial z}\right) = \frac{\gamma-1}{\gamma}\frac{\partial p'}{\partial z} - \bar{\phi}\left(h_s' - h_{\ell s}'\right) \qquad 9.3c$$

State

$$\bar{p} = \bar{\rho}\bar{T} \qquad 9.4b$$

$$\frac{p'}{\bar{p}} = \frac{\rho'}{\bar{\rho}} + \frac{T'}{\bar{T}} \qquad 9.4c$$

Droplet Dynamics

$$\bar{V}_\ell\frac{d\bar{V}_\ell}{dz} = k\left(\bar{V} - \bar{V}_\ell\right) \qquad 9.5b$$

$$\frac{\partial V_\ell'}{\partial t} + \bar{V}_\ell\frac{\partial V_\ell'}{\partial z} + V_\ell'\frac{d\bar{V}_\ell}{dz} = k\left(V' - V_\ell'\right) \qquad 9.5c$$

Droplet Enthalpy

$$\bar{h}_{\ell s} = \bar{h}_\ell + \frac{\gamma-1}{2}\bar{V}_\ell^2 = \bar{h}_{\ell s_0} \qquad 9.6b$$

$$h_{\ell s}' = h_\ell' + (\gamma-1)\bar{V}_\ell V_\ell' = h_{\ell s_0}' \qquad 9.6c$$

The foregoing set of ordinary and partial differential equations which govern the one-dimensional motion of the rocket fluid system during steady and unsteady operation, will be utilized in studying the stability of rocket motors in the intermediate frequency range where entropy waves are operative in producing combustion instability.

10. Separation of the Variables

In order to separate the variables in the governing system of linear partial differential equations, we take:

$$
\begin{aligned}
V' &= \nu(z)\, e^{st} \\
V_\ell' &= \eta(z)\, e^{st} \\
\rho' &= \delta(z)\, e^{st} \\
\rho_\ell' &= \varsigma(z)\, e^{st} \\
p' &= \varphi(z)\, e^{st} \\
\phi' &= \frac{dg}{dz}(z)\, e^{st}
\end{aligned}
\tag{10.1}
$$

Substitution into Eqs. 9.1c and 9.2c yields:

Continuity

$$\delta s + \frac{d}{dz}\left(\bar\rho \nu + \bar V \delta\right) = \frac{dg}{dz} = -s\varsigma - \frac{d}{dz}\left(\bar\rho_\ell \eta + \bar V_\ell \varsigma\right) \tag{10.2}$$

Momentum

$$s\left(\bar\rho \nu + \bar V \delta\right) + \frac{d}{dz}\left(2\bar\rho \bar V \nu + \bar V^2 \delta\right) = -\frac{1}{\gamma}\frac{d\varphi}{dz} \tag{10.3}$$

$$-s\left(\bar\rho_\ell \eta + \bar V_\ell \varsigma\right) - \frac{d}{dz}\left(2\bar\rho_\ell \bar V_\ell \eta + \bar V_\ell^2 \varsigma\right)$$

These equations are very similar to the results that would have been obtained by setting $Snh = 0$ in Eqs. 5.26 and 5.27. Before substituting into the energy Equation 9.3c, we note that it may be put into the form given by Eq. 4.10 by introducing the equation of state, i.e.

$$\frac{\partial}{\partial t}\left[p' - \bar{T}\rho' + (\gamma-1)\bar{\rho}\bar{V}V'\right] + \frac{\partial}{\partial z}\left[\bar{V}\left\{p' - \bar{T}\rho' + (\gamma-1)\bar{\rho}\bar{V}V'\right\}\right] \quad 10.4$$

$$= \frac{\gamma-1}{\gamma}\frac{\partial p'}{\partial t} + h'_{\ell s}\frac{\partial}{\partial z}(\bar{\rho}\bar{V})$$

Unlike the treatment of transverse waves in which the mixture ratio was held fixed and as a consequence $h'_{\ell s}$ was identically zero, the variation of mixture ratio is now an initial condition given by Eq. B24, while an expression for $h'_{\ell s}$ is given by Eq. B29. Accordingly, the energy equation may be written in the following form upon the introduction of Eqs. 10.1 and B29 into Eq. 10.4.

Energy

$$\frac{d}{dz}\left[\bar{V}\left\{\varphi - \delta\bar{\bar{T}} + (\gamma-1)\bar{\rho}\bar{V}\nu\right\}\right] + s\left[\frac{\varphi}{\gamma} - \delta\bar{T} + (\gamma-1)\bar{\rho}\bar{V}\nu\right] \quad 10.5$$

$$= M(\omega)\varphi_0 e^{-s\int_0^z \frac{dz'}{\bar{V}_\ell(z')}}\frac{d}{dz}(\bar{\rho}\bar{V})$$

We note that unlike Eq. 5.29 this equation has a non-homogeneous term. We will return to the energy equation later, and now let us proceed to the equation of droplet dynamics. It is:

Droplet Dynamics

$$\bar{V}_\ell\frac{d\eta}{dz} + \left(s + \frac{d\bar{V}_\ell}{dz} + k\right)\eta = k\nu \quad 10.6$$

and has the initial conditions at $z = 0$ $\begin{cases}\bar{V}_\ell(0) = \bar{V}_{\ell 0}\\ \eta(0) = \eta_0\end{cases}$

Equation 10.6 has the solution

$$\eta(z) = \frac{1}{\bar{V}_\ell} e^{-(s+k)\int_0^z \frac{dz'}{\bar{V}_\ell(z')}} \left[\int_0^z k\, \nu(z')\, e^{(s+k)\int_0^{z} \frac{dz'}{\bar{V}_\ell(z')}} dz' + \text{const.} \right]$$

and applying the boundary condition $\eta(0) = \eta_0$ we may eliminate the constant to obtain:

$$\eta(z) = \frac{k}{\bar{V}_\ell(z)} \int_0^z \nu(z') e^{(s+k)\int_z^{z'} \frac{dz''}{\bar{V}_\ell(z'')}} dz' + \eta_0 \frac{\bar{V}_{\ell 0}}{\bar{V}_\ell(z)} e^{-(s+k)\int_0^z \frac{dz'}{\bar{V}_\ell(z')}} \qquad 10.7$$

On integrating by parts, the first term yields

$$\int_0^z \nu(z') e^{(s+k)\int_z^{z'} \frac{dz''}{\bar{V}_\ell(z'')}} dz' = \frac{\nu(z)\bar{V}_\ell(z)}{s+k} - \int_0^z \left(\nu \frac{d\bar{V}_\ell}{dz} + \bar{V}_\ell \frac{d\nu}{dz} \right) \cdot$$

$$\cdot \frac{\bar{V}_\ell(z')}{(s+k)^2} \frac{d}{dz'} \left\{ e^{(s+k)\int_z^{z'} \frac{dz''}{\bar{V}_{\ell z}(z'')}} \right\} dz'$$

where we have made use of the condition $\nu(0) = 0$. If we take:

$$\frac{1}{\varphi_0}\frac{d\nu}{dz} \leq O(1) \qquad \bar{V}_\ell \leq O(M) \qquad \frac{d\bar{V}_\ell}{dz} \leq O(M) \qquad 10.8$$

then the second contribution above becomes of $O(M^2)$ and may be neglected so that:

$$\eta(z) = \frac{k\,\nu(z)}{s} + \eta_0 \frac{\bar{V}_{\ell 0}}{\bar{V}_\ell(z)} e^{-(s+k)\int_0^z \frac{dz'}{\bar{V}_\ell(z')}} \qquad 10.9$$

correct to terms of $O(M)$, and where we have not yet introduced Eq. B 22.

The initial conditions at the injector face may now be written:

$$\nu(0) = \varsigma(0) = \frac{d\varrho}{dz}(0) = 0 \qquad 10.10$$

$$\varphi(0) = \varphi_0 , \quad \eta(0) = \eta_0 = \varphi_0 \bar{V}_{\ell 0} J(\omega)$$

These conditions imply that there is no production of gas at the injector face and that the liquid density cannot vary at the injector face.

The following treatment will be seen to parallel that given for transverse waves except that S_{nh} is taken equal to zero. Combining continuity and energy, Eqs. 10.2 and 10.5 and letting

$$X(z) = \frac{\delta}{\varphi_0}(1-\bar{T}) + (\gamma-1)\bar{\rho}\bar{V}\frac{\nu}{\varphi_0} - \frac{M(w)}{w}e^{-s\int_0^z \frac{dz'}{\bar{V}_\ell(z')}}\frac{d}{dz}(\bar{\rho}\bar{V}) \qquad 10.11$$

$$Y(z) = \frac{g}{\varphi_0} - \bar{V}\frac{\varphi}{\varphi_0} + (1-\bar{\rho})\frac{\nu}{\varphi_0} - \bar{V}(1-\bar{T})\frac{\delta}{\varphi_0} - (\gamma-1)\bar{\rho}\bar{V}^2\frac{\nu}{\varphi_0}$$

we obtain:

$$\frac{d}{dz}\left(\frac{\nu}{\varphi_0}\right) + s\frac{\varphi}{\gamma\varphi_0} = -sX + \frac{dY}{dz} \qquad 10.12$$

On rearranging the momentum Equation 10.3 and taking

$$W(z) = 2\bar{\rho}\bar{V}\frac{\nu}{\varphi_0} + 2\bar{\rho}_\ell\bar{V}_\ell\frac{\eta}{\varphi_0} + \bar{V}^2\frac{\delta}{\varphi_0} + \bar{V}_\ell^2\frac{\varsigma}{\varphi_0} \qquad 10.13$$

$$Z(z) = \bar{V}\frac{\delta}{\varphi_0} - (1-\bar{\rho})\frac{\nu}{\varphi_0} + \bar{\rho}_\ell\frac{\eta}{\varphi_0} + \bar{V}_\ell\frac{\varsigma}{\varphi_0}$$

we obtain:

$$\frac{d}{dz}\left(\frac{\varphi}{\gamma\varphi_0}\right) + s\frac{\nu}{\varphi_0} = -sZ - \frac{dW}{dz} \qquad 10.14$$

Adding and subtracting terms in Eqs. 10.12 and 10.14 we may write:

$$\frac{d}{dz}\left(\frac{\nu}{\varphi_0} - Y\right) + s\left(\frac{\varphi}{\gamma\varphi_0} + W\right) = s(W-X) \qquad 10.15$$

$$\frac{d}{dz}\left(\frac{\varphi}{\gamma\varphi_0} + W\right) + s\left(\frac{\nu}{\varphi_0} - Y\right) = -s(Y+Z) \qquad 10.16$$

For convenience we let:

$$A(z) = \left(\frac{\nu}{\varphi_0} - Y\right), \quad E(z) = (W-X) \qquad 10.17$$

$$B(z) = \left(\frac{\varphi}{\gamma\varphi_0} + W\right), \quad F(z) = (Y+Z)$$

and then Eqs. 10.15 and 10.16 become:

$$\frac{dA}{dz} + sB = sE \qquad 10.18$$

$$\frac{dB}{dz} + sA = -sF \qquad 10.19$$

We may eliminate $B(z)$ by differentiating Eq. 10.18 and combining it with Eq. 10.19:

$$\frac{d^2A}{dz^2} - s^2 A = h \qquad 10.20$$

where

$$h = s\frac{dE}{dz} + s^2 F \qquad 10.21$$

Since the right hand side of Eq. 10.20 is not given explicitly, use of the method of variation of parameters yields:

$$A(z) = -C_1 \cosh sz - C_2 \sinh sz + \frac{1}{s}\int_0^z h(z')\sinh s(z-z')\,dz' \qquad 10.22$$

which may be compared with Eq. 5.46 after setting $Snh = 0$ in the latter. Substituting for $h(z')$ from Eq. 10.21, we may integrate Eq. 10.22 by parts. We make use of the condition

$$E(0) = W(0) - X(0) = 2\bar{\rho}_{l0}\bar{V}_{l0}^2 J(w) + \frac{M(w)}{S}\left(\frac{d\bar{V}}{dz}\right)_0$$

which may, however, be absorbed into the coefficient of $\sinh sz$ and then we obtain:

$$A(z) = \frac{\nu}{\varphi_0} - Y = -C_1 \cosh sz - C_2 \sinh sz \qquad 10.23$$

$$+ s\int_0^z E(z')\cosh s(z-z')\,dz'$$

$$+ s\int_0^z F(z')\sinh s(z-z')\,dz'$$

Upon differentiating Eq. 10.23 and substituting back into Eq. 10.18, there is obtained:

$$B(z) = \frac{\varphi}{\gamma\varphi_0} + W = C_1 \cosh sz + C_2 \sinh sz \qquad 10.24$$

$$-s\int_0^z E(z')\cosh s(z-z')\,dz'$$

$$-s\int_0^z F(z')\cosh s(z-z')\,dz'$$

Let us determine the constants C_1 and C_2 by introducing the initial conditions at $z = 0$. From Eqs. 10.11 and 10.13, there is obtained:

$$Y(0) = 0 \tag{10.25}$$

$$W(0) = 2\bar{\rho}_{l0}\bar{V}_{l0}\frac{\eta_0}{\varphi_0} = 2\bar{\rho}_{l0}\bar{V}_{l0}^2 J(w)$$

and hence simultaneous solution of Eqs. 10.23 and 10.24 at $z = 0$ yields:

$$C_1 = 0$$
$$C_2 = \frac{1}{\gamma s} + 2\bar{\rho}_{l0}\bar{V}_{l0}^2 J(w) \tag{10.26}$$

and now substituting these results into Eqs. 10.23 and 10.24 we finally obtain:

$$\frac{\gamma\nu}{\varphi_0} = \gamma Y - \left[1 + 2\gamma\bar{\rho}_{l0}\bar{V}_{l0}^2 J(w)\right]\sinh sz \tag{10.27}$$
$$+ s\int_0^z \gamma E(z')\cosh s(z-z')\,dz'$$
$$+ s\int_0^z \gamma F(z')\sinh s(z-z')\,dz'$$

and

$$\frac{\varphi}{\varphi_0} = -\gamma W + \left[1 + 2\gamma\bar{\rho}_{l0}\bar{V}_{l0}^2 J(w)\right]\cosh sz \tag{10.28}$$
$$- s\int_0^z \gamma E(z')\sinh s(z-z')\,dz'$$
$$- s\int_0^z \gamma F(z')\cosh s(z-z')\,dz'$$

These two integral equations will be used in the solution of the problem. We must also obtain the equation for the entropy variation in the gas before we can write down the characteristic equation for the chamber. From Eq. 5.59 we have:

$$\bar{p}\,\frac{\epsilon(z)}{\varphi_0} = \frac{1}{\gamma}\,\frac{\varphi(z)}{\varphi_0} - \frac{\delta(z)}{\varphi_0}\,\bar{T} \qquad 10.29$$

Rewriting the energy equation 10.5 we have:

$$\frac{d}{dz}\left[\bar{V}\left\{\frac{\varphi}{\varphi_0} - \frac{\delta}{\varphi_0}\bar{T} + (\gamma-1)\bar{\rho}\bar{V}\frac{\nu}{\varphi_0}\right\}\right] + \frac{s}{\bar{V}}\left[\bar{V}\left\{\frac{\varphi}{\varphi_0} - \frac{\delta}{\varphi_0}\bar{T} + (\gamma-1)\bar{\rho}\bar{V}\frac{\nu}{\varphi_0}\right\}\right] \qquad 10.30$$
$$= \frac{\gamma-1}{\gamma}\,s\,\frac{\varphi}{\varphi_0} + M(w)\,e^{-s\int_0^z \frac{dz'}{\bar{V}(z')}}\,\frac{d}{dz}\left(\bar{\rho}\bar{V}\right)$$

Integrating Eq. 10.30 and combining the result with Eq. 10.29,

$$\frac{\gamma\epsilon}{\varphi_0} = -(\gamma-1)\left\{\bar{\rho}\bar{V}\,\frac{\gamma\nu}{\varphi_0} + \frac{1}{\bar{V}}\int_0^z e^{s\int_z^{z'}\frac{dz''}{\bar{V}(z'')}}\,d\left[\bar{V}(z')\,\frac{\varphi(z')}{\varphi_0}\right]\right\} \qquad 10.31$$
$$+ \frac{\gamma M(w)}{\bar{V}}\int_0^z e^{-s\int_0^{z'}\frac{dz''}{\bar{V}_\ell(z'')}}\,e^{s\int_z^{z'}\frac{dz''}{\bar{V}(z'')}}\,\frac{d\bar{V}}{dz'}(z')\,dz'$$

so that upon expanding the term in brackets, we finally obtain:

$$\frac{\gamma\epsilon}{\varphi_0} = -(\gamma-1)\left\{\bar{\rho}\bar{V}\,\frac{\gamma\nu}{\varphi_0} + \frac{1}{\bar{V}}\int_0^z e^{s\int_z^{z'}\frac{dz''}{\bar{V}(z'')}}\,\frac{\varphi(z')}{\varphi_0}\,\frac{d\bar{V}}{dz'}(z')\,dz'\right. \qquad 10.32$$
$$\left. + \frac{1}{\bar{V}}\int_0^z e^{s\int_z^{z'}\frac{dz''}{\bar{V}(z'')}}\,\bar{V}(z')\,\frac{d}{dz'}\left(\frac{\varphi(z')}{\varphi_0}\right)dz'\right\}$$
$$+ \frac{\gamma M(w)}{\bar{V}}\int_0^z e^{-s\int_0^{z'}\frac{dz''}{\bar{V}_\ell(z'')}}\,e^{s\int_z^{z'}\frac{dz''}{\bar{V}(z'')}}\,\frac{d\bar{V}}{dz'}(z')\,dz'$$

where for one-dimensional flow:

$$S(z,t) = \epsilon(z)\,e^{st} \qquad 10.33$$

11. The Burning Rate Perturbation

We will now derive a relation for the burning rate perturbation when the mixture ratio oscillates. Following Crocco and Cheng, the total time lag τ_t is taken as the sum of a space varying insensitive part $\bar{\tau}_i$, and a time and space varying sensitive part τ.

$$\tau_t(z,t) = \bar{\tau}_t(z) + \tau(z,t) \qquad 11.1$$

where τ is a function of the interaction indices characteristic of the propellant combination. The interaction indices may be discussed in terms of the functional dependence of the factors controlling the rates of the conditioning processes. It is expected that the conditioning processes will, in the case of a bipropellant rocket, depend to some degree on the mixture ratio r. This may be expressed mathematically as follows. The overall rate of the processes at a given location are a function $f(p,T,r,y)$ of pressure, temperature, mixture ratio and any other physical factor y. We may expand this function of several variables in a Taylor series about the steady state operating condition where the local values of the factors are $\bar{p}$, $\bar{T}$, $\bar{r}$ and $\bar{y}$. Thus, applying small perturbations p', T', r' and y' we obtain for the new process rate:

$$f(p,T,r,y) = f(\bar{p}+p', \bar{T}+T', \bar{r}+r', \bar{y}+y') \qquad 11.2$$

$$= \overline{f(p,T,r,y)} + p'\overline{\frac{\partial f}{\partial p}} + T'\overline{\frac{\partial f}{\partial T}} + r'\overline{\frac{\partial f}{\partial r}} + y'\overline{\frac{\partial f}{\partial y}}$$

where the barred quantities are to be evaluated at $p=\bar{p}$, $T=\bar{T}$, $r=\bar{r}$ and $y=\bar{y}$. Let us now assume that the temperature and the physical factors are correlated to the pressure and mixture ratio, i.e. $T=T(p,r)$, $y=y(p,r)$ and there follows:

$$f = \bar{f}\left(1 + n_1\frac{p'}{\bar{p}} + n_2\frac{r'}{\bar{r}}\right) \qquad 11.3$$

where we have introduced the constants:

$$n_1 = \frac{\bar{p}}{\bar{f}}\overline{\left(\frac{\partial f}{\partial p} + \frac{\partial T}{\partial p}\frac{\partial f}{\partial T} + \frac{\partial y}{\partial p}\frac{\partial f}{\partial y}\right)} \qquad 11.4$$

$$n_2 = \frac{\bar{r}}{\bar{f}}\overline{\left(\frac{\partial f}{\partial r} + \frac{\partial T}{\partial r}\frac{\partial f}{\partial T} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y}\right)}$$

Now the definition of the sensitive time lag is:

$$\int_{t-\tau}^{t} f(t')\,dt' = E_a \tag{11.5}$$

where E_a represents the quantity of energy required to initiate burning at station z and time t. The integral must be evaluated following the motion of the propellant element, however, since the properties of the propellant element now depend on the mixture ratio, we observe that E_a is no longer a constant, but varies with the mixture ratio according to:

$$E_a = \bar{E}_a \left(1 + n_3 \frac{r'}{\bar{r}} \right) \tag{11.6}$$

where

$$n_3 = \overline{\frac{\partial E_a}{\partial r}}\,\frac{\bar{r}}{\bar{E}_a} \tag{11.7}$$

Since E_a is associated with a particular particle, it follows that the perturbation $\frac{r'}{\bar{r}}$ must be evaluated at the instant of injection of that particle. Then introducing Eqs. 11.3 and 11.6 into Eq. 11.5, and evaluating at time t:

$$\int_{t-\tau}^{t} \bar{f}\left[z'(t'),t'\right]\left\{1 + n_1 \frac{p'}{\bar{p}}\left[z'(t'),t'\right] + n_2 \frac{r'}{\bar{r}}\left[z'(t'),t'\right]\right\}dt' \tag{11.9}$$
$$= \bar{E}_a(\bar{z})\left\{1 + n_3 \left(\frac{r'}{\bar{r}}\right)_0 \left[t - \tau_t(z,t)\right]\right\}$$

Since $\frac{r'}{\bar{r}}\left[z(t'),t'\right]$ does not change during the sensitive time lag, but rather it retains the value it had at the instant of injection, it may be taken outside the integral, so that:

$$\int_{t-\tau}^{t} n_2 \frac{r'}{\bar{r}}\left[z'(t'),t'\right]dt' = n_2 \left(\frac{r'}{\bar{r}}\right)_0 \left[t' - \int_0^{z'} \frac{dz''}{V_\ell\left[z'',t''(z'')\right]}\right] \int_{t-\tau}^{t} dt' \tag{11.10}$$
$$= n_2\,\bar{\tau}(\bar{z}) \left(\frac{r'}{\bar{r}}\right)_0 \left[t' - \int_0^{z'} \frac{dz''}{V_\ell\left[z'',t''(z'')\right]}\right]$$

upon neglecting higher order terms. Substituting back into Eq. 11.9 we find:

$$\tau(z,t) - \bar{\tau}(\bar{z}) = \bar{\tau}(\bar{z})\left\{(n_3 - n_2)\left(\frac{r'}{\bar{r}}\right)_0 [t - \tau_t]\right\} - \int_{t-\tau}^{t} n_1 \frac{p'}{\bar{p}} [z'(t'), t'] dt' \qquad 11.11$$

Now setting:

$$n = n_1$$
$$m = n_3 - n_2 \qquad 11.12$$

we identify the latter as the interaction indices and obtain:

$$\tau(z,t) - \bar{\tau}(\bar{z}) = -n \int_{t-\tau(z,t)}^{t} \frac{p'}{\bar{p}} [z'(t'), t'] dt' + m\bar{\tau}(\bar{z})\left(\frac{r'}{\bar{r}}\right)_0 [t - \tau_t] \qquad 11.13$$

This equation reflects the change in the sensitive time lag from its steady state value $\bar{\tau}(\bar{z})$ for a particle which begins burning precisely at time t and station z, when it has been injected with mixture ratio perturbation $r'_0[t - \tau_t(z,t)]$ at time $t - \tau_t(z,t)$ and has traveled through the chamber with velocity $V_\ell'[z'(t'), t']$ and has been exposed to pressure perturbations $p'[z'(t'), t']$.

Differentiating Eq. 11.13 with respect to t and neglecting higher order terms, there results:

$$\frac{d\tau}{dt}(z,t) = -\frac{n}{\bar{p}}\left\{p'[z,t] - p'[z'(t-\tau), t - \tau(z,t)]\right\} + m\bar{\tau}(\bar{z})\frac{d}{dt}\left(\frac{r'}{\bar{r}}\right)_0 [t - \tau_t] \qquad 11.14$$

Following the treatment in Section 6, we observe that for one-dimensional flow Eq. 6.17 becomes:

$$\delta\dot{m}_b(z,t) = \delta\dot{m}_i[t - \tau_t(z,t)]\left\{1 - \frac{d\tau}{dt}(z,t)\right\} \qquad 11.15$$

where $\delta \dot{m}_i$ and $\delta \dot{m}_b$ are the fractional injection rate and fractional burning rate respectively. Since the injection rate is no longer a constant, Eq. 6.19 does not hold, and instead we consider $\delta \dot{m}_i$ to be the same geometric fraction of $\dot{m}_i$ as $\overline{\delta \dot{m}_i}$ is of $\overline{\dot{m}_i}$, that is:

$$\frac{\delta \dot{m}_i}{\dot{m}_i} = \frac{\overline{\delta \dot{m}_i}}{\overline{\dot{m}_i}} \qquad 11.16$$

Utilizing Eq. 6.18, we have

$$\delta \dot{m}_i = \frac{\dot{m}_i}{\overline{\dot{m}_i}} \overline{\delta \dot{m}_i} = \left\{ 1 + \left(\frac{\dot{m}_i'}{\overline{\dot{m}_i}} \right) \right\} \overline{\delta \dot{m}_b} \qquad 11.17$$

and then Eq. 11.15 becomes:

$$\delta \dot{m}_b (z,t) = \overline{\delta \dot{m}_b} (\bar{z}) \left\{ 1 + \left(\frac{\dot{m}_i'}{\overline{\dot{m}_i}} \right) \left[t - \tau_t(z,t) \right] - \frac{d\tau}{dt}(z,t) \right\} \qquad 11.18$$

upon neglecting higher order terms.

Introducing ϕ as the instantaneous rate per unit volume at which gas is produced in the chamber, we may write

$$\delta \dot{m}_b (z,t) = \phi(z,t)\, dA\, dz \qquad 11.19$$

by definition of the fractional burning rate. In the steady state, this reduces to:

$$\overline{\delta \dot{m}_b} (\bar{z}) = \bar{\phi}(\bar{z})\, dA\, d\bar{z} \qquad 11.20$$

Hence, combining Eqs. 11.18, 11.19 and 11.20 we find:

$$\phi(z,t)\, dz = \bar{\phi}(\bar{z})\, d\bar{z} \left\{ 1 + \left(\frac{\dot{m}_i'}{\overline{\dot{m}_i}} \right) \left[t - \tau_t(z,t) \right] - \frac{d\tau}{dt}(z,t) \right\} \qquad 11.21$$

This equation relates the rate of burned gas generation at a given instant of time t and location z to the steady state gas generation at location $\bar{z}$, the variation in the time lag, and the oscillating injection rate. The effect of the oscillating mixture ratio is of course included in $\frac{d\tau}{dt}$.

Separation of the variables enables us to write:

$$\phi(z,t) = \bar{\phi}(z) + \frac{dg}{dz}(z) e^{st} \qquad 11.22$$

where $\quad \bar{\Phi}(\bar{z}) = \dfrac{d\bar{V}}{d\bar{z}}$

Thus Eq. 11.21 becomes:

$$\frac{d\bar{V}}{dz}(z)\,dz + \frac{dq}{dz}(z)e^{st}\,dz = \frac{d\bar{V}}{d\bar{z}}(\bar{z})\,d\bar{z} \qquad 11.23$$

$$+ \frac{d\bar{V}}{d\bar{z}}(\bar{z})\left(\frac{\dot{m}_i'}{\bar{\dot{m}}_i}\right)[t-\tau_t(z,t)]\,d\bar{z} - \frac{d\bar{V}}{d\bar{z}}(\bar{z})\,\frac{d\tau}{dt}(z,t)\,d\bar{z}$$

Before integrating this equation, let us examine the expressions for $\dfrac{d\tau}{dt}(z,t)$ and $\left(\dfrac{\dot{m}_i'}{\bar{\dot{m}}_i}\right)[t-\tau_t]$. Starting with Eqs. 11.14 and B 24, we obtain upon neglecting higher order terms:

$$\frac{d\tau}{dt}(z,t) = -\frac{n}{\bar{p}}\left\{p'(\bar{z}) - p'[\bar{\xi}, t-\bar{\tau}(\bar{z})]\right\} \qquad 11.24$$

$$+ n\,\bar{\tau}(\bar{z})\frac{d}{dt}\left(\frac{p'}{\bar{p}}\right)_0[t-\bar{\tau}_t(\bar{z})] + \text{higher order terms}$$

Similarly, we may evaluate the mass flow perturbation as:

$$\left(\frac{\dot{m}_i'}{\bar{\dot{m}}_i}\right)[t-\tau_t(z,t)] \cong \left(\frac{\dot{m}'}{\bar{\dot{m}}_i}\right)[t-\bar{\tau}_t(\bar{z})] \qquad 11.25$$

Substituting back into Eq. 11.23 we obtain:

$$\frac{d\bar{V}}{dz}(z)\,dz + \frac{dq}{dz}(z)e^{st}\,dz = \frac{d\bar{V}}{d\bar{z}}(\bar{z})\,d\bar{z} \qquad 11.26$$

$$+ \frac{d\bar{V}}{d\bar{z}}(\bar{z})\left(\frac{\dot{m}_i'}{\bar{\dot{m}}_i}\right)[t-\bar{\tau}_t(\bar{z})]\,d\bar{z} + \frac{n}{\bar{p}}\frac{d\bar{V}}{d\bar{z}}(\bar{z})\left\{p'(\bar{z}) - p'[\bar{\xi}, t-\bar{\tau}(\bar{z})]\right\}d\bar{z}$$

$$- n\,\frac{d\bar{V}}{d\bar{z}}(\bar{z})\,\bar{\tau}(\bar{z})\,\frac{d}{dt}\left(\frac{p'}{\bar{p}}\right)_0[t-\bar{\tau}_t(\bar{z})] + \text{higher order terms}$$

And now let us integrate each term from zero to the appropriate upper limit, i.e. z or $\bar{z}$, while keeping the time fixed.

$$\int_0^z \frac{dq}{dz'}(z')e^{st}\,dz' = -\int_0^z \frac{d\bar{V}}{dz'}(z')dz' + \int_0^z \frac{d\bar{V}}{dz'}(z')dz' \qquad 11.27$$

$$+ \int_0^{\bar{z}} \left(\frac{\dot{m}_i'}{\bar{\dot{m}}_i}\right)[t-\tau_t(z')]\frac{d\bar{V}}{dz'}(z')dz' + \int_0^{\bar{z}} \frac{n}{\bar{p}}\left\{p'(z') - p'[\bar{\xi}, t-\bar{\tau}(z')]\right\}\frac{d\bar{V}}{dz'}(z')dz'$$

$$- \int_0^{\bar{z}} m\bar{\tau}(z')\frac{d}{dt}\left(\frac{r'}{\bar{r}}\right)_0 [t-\bar{\tau}_t(z')]\frac{d\bar{V}}{dz'}(z')dz'$$

Noting that $\bar{V}(0) = q(0) = 0$ we obtain:

$$q(z)e^{st} = -\left(\bar{V}(z) - \bar{V}(\bar{z})\right) + \int_0^{\bar{z}} \left(\frac{\dot{m}_i'}{\bar{\dot{m}}_i}\right)[t-\bar{\tau}_t(z')]\frac{d\bar{V}}{dz'}(z')dz' \qquad 11.28$$

$$+ \int_0^{\bar{z}} \frac{n}{\bar{p}}\left\{p'(z') - p'[\bar{\xi}(z'), t-\bar{\tau}(z')]\right\}\frac{d\bar{V}}{dz'}(z')dz'$$

$$- \int_0^{\bar{z}} m\bar{\tau}(z')\frac{d}{dt}\left(\frac{r'}{\bar{r}}\right)_0 [t-\bar{\tau}_t(z')]\frac{d\bar{V}}{dz'}(z')dz'$$

We may eliminate the term $\bar{V}(z) - \bar{V}(\bar{z})$ by a Taylor expansion, see Eq. 6.47, if we can first determine $z - \bar{z}$. Since Eq. 11.5 may also be written:

$$\int_{\xi}^{z} \frac{f[z', t'(z')]\,dz'}{V_\ell[z', t'(z')]} = E_a(z) \qquad 11.29$$

where ξ is the spatial location at which an element burning at z enters its sensitive time lag, we obtain for the steady state:

$$\int_{\bar{\xi}}^{\bar{z}} \frac{\bar{f}(z')\,dz'}{\bar{V}_\ell(z')} = \bar{E}_a(\bar{z}) \qquad 11.30$$

But $\bar{f}(z')$ is a constant, and hence we may split the integral in Eq. 11.30 into three parts:

$$\bar{f}\left\{\int_{\bar{\xi}}^{\xi} \frac{dz'}{\bar{V}_\ell(z')} + \int_{\xi}^{z} \frac{dz'}{\bar{V}_\ell(z')} + \int_{z}^{\bar{z}} \frac{dz'}{\bar{V}_\ell(z')}\right\} = \bar{E}_a(\bar{z}) \qquad 11.31$$

Upon neglecting higher order terms, we may write:

$$\int_{\bar{\xi}}^{\xi} \frac{dz'}{\bar{V}_\ell(z')} = \frac{\xi(z,t) - \bar{\xi}(\bar{z})}{\bar{V}_\ell(\xi)} \; ; \; \int_{z}^{\bar{z}} \frac{dz'}{\bar{V}_\ell(z')} = \frac{\bar{z} - z}{\bar{V}_\ell(z)} \qquad 11.32$$

and substituting back into Eq. 11.31 we obtain:

$$\bar{f}\left\{ \frac{\xi(z,t) - \bar{\xi}(\bar{z})}{\bar{V}_\ell(\xi)} + \int_{\xi}^{z} \frac{dz'}{\bar{V}_\ell(z')} + \frac{\bar{z} - z}{\bar{V}_\ell(z)} \right\} = \bar{E}_a(\bar{z}) \qquad 11.33$$

Now Eq. 11.8 may be rewritten in the form:

$$\bar{f} \int_{\xi}^{z} \left\{ 1 + n_1 \frac{p'}{\bar{p}}[z', t'(z')] + n_2 \frac{\mu'}{\bar{\mu}}[z', t'(z')] \right\} \frac{dz'}{V_\ell[z', t'(z')]} \qquad 11.34$$

$$= \bar{E}_a(\bar{z}) \left\{ 1 + n_3 \left(\frac{\mu'}{\bar{\mu}}\right)_0 [t - \tau_t(z,t)] \right\}$$

and then combining Eqs. 11.33 and 11.34 and rearranging, we find:

$$\frac{z - \bar{z}}{\bar{V}_\ell(z)} = \frac{\xi(z,t) - \bar{\xi}(\bar{z})}{\bar{V}_\ell(\xi)} + \int_{\xi}^{z} \frac{dz'}{\bar{V}_\ell(z')}$$

$$- \left\{ 1 - n_3 \left(\frac{\mu'}{\bar{\mu}}\right)_0 [t - \tau_t(z,t)] \right\} \left\{ \int_{\xi}^{z} \left(1 + n_1 \frac{p'}{\bar{p}}[z', t'(z')] \right.\right. \qquad 11.35$$

$$\left.\left. + n_2 \frac{\mu'}{\bar{\mu}}[z', t'(z')] - \frac{V_\ell'}{\bar{V}_\ell(z')}[z', t'(z')] \right) \frac{dz'}{\bar{V}_\ell(z')} \right\}$$

where we have made use of: $\dfrac{1}{V_\ell[z', t'(z')]} \cong \left[1 - \dfrac{V_\ell'}{\bar{V}_\ell(z')}[z', t'(z')] \right] \dfrac{1}{\bar{V}_\ell(z')}$.

The term involving $\xi - \bar{\xi}$ may be eliminated by introducing the definition of the insensitive time lag:

$$\int_0^{\xi} \frac{dz'}{V_\ell[z', t'(z')]} = \bar{\tau}_i = \int_0^{\bar{\xi}} \frac{dz'}{\bar{V}_\ell(z')} \qquad 11.36$$

On splitting the second integral into two parts, there follows:

$$\frac{\xi(z,t) - \bar{\xi}}{\bar{V}_\ell(\xi)} \cong \int_0^{\xi} \frac{dz'}{\bar{V}_\ell(z')} - \int_0^{\xi} \frac{dz'}{V_\ell[z', t'(z')]} \cong \int_0^{\xi} \frac{V_\ell'[z', t'(z')]\,dz'}{\bar{V}_\ell^{\,2}(z')} \qquad 11.37$$

Substituting back into Eq. 11.35, utilizing the definitions Eq. 11.12, and combining terms, we finally derive:

$$\frac{z-\bar{z}}{\bar{V}_\ell(z)} = \int_0^z \frac{V_\ell'[z',t'(z')]\,dz'}{\bar{V}_\ell^2(z')} - n\int_\xi^z \frac{p'[z',t'(z')]dz'}{\bar{p}\,\bar{V}_\ell(z')} \qquad 11.38$$

$$+ m\left(\frac{\mu'}{\bar{\mu}}\right)_0 [t-\tau_t(z,t)]\int_\xi^z \frac{dz'}{\bar{V}_\ell(z')}$$

This equation yields the shift in the burning station from the steady state location $\bar{z}$ for a particular propellant element injected at time $t-\tau_t$. There are three contributions to this shift. The first term on the right hand side gives the effect of a change in droplet velocity, while the second and third terms correspond to a change in time lag due to pressure oscillations and mixture ratio perturbation.

Introducing Eq. 11.38 into Eq. 6.47 there results:

$$\bar{V}(z) - \bar{V}(\bar{z}) = \frac{d\bar{V}}{d\bar{z}}(\bar{z})\,\bar{V}_\ell(z)\left\{\int_0^z \frac{V_\ell'[z',t'(z')]\,dz'}{\bar{V}_\ell^2(z')}\right. \qquad 11.39$$

$$\left. - n\int_\xi^z \frac{p'[z',t'(z')]\,dz'}{\bar{p}\,\bar{V}_\ell(z')} + m\left(\frac{\mu'}{\bar{\mu}}\right)_0 [t-\tau_t(z,t)]\int_\xi^z \frac{dz'}{\bar{V}_\ell(z')}\right\}$$

and now substituting this result into Eq. 11.28, we obtain the form:

$$q(z)e^{st} = \int_0^{\bar{z}} \left[n\left\{\frac{p'}{\bar{p}}(z',t) - \frac{p'}{\bar{p}}\left[\bar{\xi}(z'), t-\bar{\tau}(z')\right]\right\} + \left(\frac{\dot{m}_i}{\bar{\dot{m}}_i}\right)_0 [t-\tau_t(z')]\right.$$

$$\left. - m\bar{\tau}(z')\frac{d}{dt}\left(\frac{\mu'}{\bar{\mu}}\right)_{..}[t-\bar{\tau}_t(z')]\right]\frac{d\bar{V}}{dz'}(z')dz' \qquad 11.40$$

$$+ \bar{V}_\ell(z)\frac{d\bar{V}}{dz}(z)\left\{\int_{\bar{\xi}}^z \frac{n\,p'[z',t'(z')]\,dz'}{\bar{p}\,V_{\ell z}(z')} - m\bar{\tau}(z)\left(\frac{\mu'}{\bar{\mu}}\right)_0[t-\bar{\tau}_t(z)]\right.$$

$$\left. - \int_0^z \frac{V_\ell'[z',t'(z')]\,dz'}{\bar{V}_\ell^2(z')}\right\}$$

On examining this result, we observe that the perturbation in the gas flow consists of the contributions of two groups, each consisting of three terms.

The first group of terms may be called the time-wise contribution, and represents the effect of the perturbations in the local burning rate, which in turn is due to the variation in the time lag due to the pressure oscillations, the initial perturbation in mass flow and the time rate of variation of the mixture ratio. The second group of terms, which may be called the space-wise contribution, represents the effect of the displacement of the location where a given propellant element burns.

And now using Eqs. 10.1, 10.9 and the results of Appendix B, and then dividing by $\varphi_o e^{st}$, we have:

$$
\begin{aligned}
q(z) = {} & n \int_0^{\bar{z}} \frac{1}{\bar{p}} \left[\frac{\varphi(z')}{\varphi_o} - \frac{\varphi}{\varphi_o} \left[\bar{\xi}(z') \right] e^{-s\bar{\tau}(z')} \right] \frac{d\bar{V}}{dz'}(z')\,dz' \\
& + \int_0^{\bar{z}} \left\{ H(w) - m\bar{\tau}(z')\, s\, G(w) \right\} e^{-s\bar{\tau}_t(z')} \frac{d\bar{V}}{dz'}(z')\,dz' \\
& + \bar{V}_\ell(z) \frac{d\bar{V}}{dz}(z) \left\{ n \int_{\bar{\xi}}^{z} \frac{\varphi(z')}{\bar{p}\varphi_o} \frac{e^{s\int_z^{z'} \frac{dz''}{\bar{V}_\ell(z'')}}}{\bar{V}_\ell(z')}\,dz' - m\bar{\tau}(z) G(w) e^{-s\bar{\tau}_t(z)} \right. \\
& \left. - \int_0^{z} \left[\frac{k}{s} \frac{\nu(z')}{\varphi_o} \frac{1}{\bar{V}_\ell^{2}(z')} + J(w) \frac{\bar{V}_{\ell o}^{2}}{\bar{V}_\ell^{3}(z')} e^{-(s+k)\int_0^{z'} \frac{dz''}{\bar{V}_\ell(z'')}} \right] e^{s\int_z^{z'} \frac{dz''}{\bar{V}_\ell(z'')}}\,dz' \right\}
\end{aligned}
\qquad 11.41
$$

These terms may be simplified by considering an order of magnitude analysis of the six parts which contribute to the source term. The maximum local value of $\frac{d\bar{V}}{dz}$ is assumed to be O(1), and in addition, it is assumed that

$$
\frac{\varphi}{\varphi_o}, \quad \frac{\delta}{\varphi_o}, \quad \frac{\nu}{\varphi_o}, \quad \frac{1}{\varphi_o}\frac{d\varphi}{dz}, \quad \frac{1}{\varphi_o}\frac{d\nu}{dz} = O(1) \qquad 11.42
$$

We also note that if $G(w)$, $H(w)$, $J(w)$ and $M(w)$ are each O(1), then Appendix B shows that $\frac{\Gamma'}{\Gamma}$, $\frac{\dot{m}_i'}{\dot{m}_i}$, $\frac{V_{\ell o}'}{V_{\ell o}}$ and $h_{\ell s}'$ are each of the order of a perturbation, which is in agreement with our understanding of these terms.

Since $\int_0^z \frac{d\bar{V}}{dz'}(z')dz'$ must integrate to a quantity of O(M), no matter how large $\frac{d\bar{V}}{dz}(z')$ is locally, on taking a Taylor expansion in $\frac{\varphi(z')}{\varphi_0}$, we may show that the first term in Eq. 11.41 becomes:

$$n \int_0^z \frac{\varphi(z')}{\varphi_0}\left[1 - e^{-s\bar{\tau}(z')}\right]\frac{d\bar{V}}{dz'}(z')dz' + O(M^2) \qquad 11.43$$

where the integral yields a term of O(M). The second term in Eq. 11.41 is:

$$\int_0^{\bar{z}}\left\{H(w) - m\bar{\tau}(z')sG(w)\right\}e^{-s\bar{\tau}t(z')}\frac{d\bar{V}}{dz'}(z')dz' = \int_0^z\left\{\qquad\right\}e^{-s\bar{\tau}t(z')}\frac{d\bar{V}}{dz'}(z')dz' + O(M^2)$$

and integrates to O(M) and cannot be simplified. Noting that:

$$\int_{\xi}^{\bar{z}} e^{s\int_z^{z'}\frac{dz''}{\bar{V}_\ell(z'')}}\cdot\frac{s\,dz'}{\bar{V}_\ell(z')} \equiv 1 - e^{-s\int_{\xi}^{\bar{z}}\frac{dz'}{\bar{V}_\ell(z')}} \cong 1 - e^{-s\bar{\tau}(z)}$$

the third contribution to the source term becomes:

$$\bar{V}_\ell(z)\frac{d\bar{V}}{dz}(z)\frac{n}{s}\frac{\varphi(z)}{\varphi_0}\left[1 - e^{-s\bar{\tau}(z)}\right] + O(M^2) \qquad 11.44$$

which is of O(M). The fourth terms is:

$$\bar{V}_\ell(z)\frac{d\bar{V}(z)}{dz}m\,\bar{\tau}(z)G(w)e^{-s\bar{\tau}t(z')} = O(M)$$

On integrating the fifth term by parts we obtain:

$$\bar{V}_\ell(z)\frac{d\bar{V}(z)}{dz}\frac{k}{s}\int_0^z\frac{\nu(z')}{\varphi_0}\frac{1}{\bar{V}_\ell^2(z')}e^{s\int_z^{z'}\frac{dz''}{\bar{V}_\ell(z'')}}dz' =$$

$$\frac{k}{s^2}\frac{\nu(z)}{\varphi_0}\frac{d\bar{V}(z)}{dz} - \bar{V}_\ell(z)\frac{d\bar{V}(z)}{dz}\frac{k}{s^3}\cdot\overline{V_\ell(z')\frac{d}{dz'}\left(\frac{\nu(z')}{\bar{V}_\ell(z')\varphi_0}\right)}\int_0^z\frac{d}{dz'}\left[e^{s\int_z^{z'}\frac{dz''}{\bar{V}_\ell(z'')}}\right]dz'$$

where the bar represents a proper mean value so that when $s \cong O(1)$, the fifth term becomes:

$$\frac{k}{s^2}\frac{\nu(z)}{\varphi_0}\frac{d\bar{V}}{dz}(z) + O(M^2)$$

And now taking the last term, we have:

$$\bar{V}_\ell(z)\frac{d\bar{V}}{dz}(z)\int_0^z J(w)\frac{\bar{V}_{\ell 0}^2}{\bar{V}_\ell^3(z')}e^{-(s+k)\int_0^{z'}\frac{dz''}{\bar{V}_\ell(z'')}}e^{s\int_z^{z'}\frac{dz''}{\bar{V}_\ell(z'')}}dz'$$

$$= \bar{V}_\ell(z)\frac{d\bar{V}(z)}{dz}J(w)\bar{V}_{\ell 0}^2 e^{-s\int_0^z\frac{dz'}{\bar{V}_\ell(z')}}\int_0^z\frac{e^{-k\int_0^{z'}\frac{dz''}{\bar{V}_\ell(z'')}}}{\bar{V}_\ell^3(z')}dz' \qquad 11.45$$

which is of O(1).

Substituting these results back into Eq. 11.41, we have:

$$\frac{q(z)}{\varphi_0} = n\,Q(z) + \int_0^z \left\{H(w) - m\,\bar{\tau}(z')\,s\,G(w)\right\} e^{-s\bar{\tau}_t(z')} \frac{d\bar{V}}{dz'}(z')\,dz'$$
$$+ \bar{V}_\ell(z)\frac{d\bar{V}}{dz}(z)\,\frac{n}{s}\,\frac{\varphi(z)}{\varphi_0}\left[1 - e^{-s\bar{\tau}(z')}\right] \qquad 11.46$$
$$- \bar{V}_\ell(z)\frac{d\bar{V}}{dz}(z)\,m\,\bar{\tau}(z)\,G(w)\,e^{-s\bar{\tau}_t(z)} - \frac{k}{s^2}\,\frac{\nu(z)}{\varphi_0}\,\frac{d\bar{V}}{dz}(z)$$
$$- \bar{V}_\ell(z)\frac{d\bar{V}}{dz}(z)\,J(w)\,\bar{V}_{\ell 0}^{\,2}\,e^{-s\int_0^z \frac{dz'}{\bar{V}_\ell(z')}}\int_0^z \frac{e^{-k\int_0^{z'}\frac{dz''}{\bar{V}_\ell(z'')}}}{\bar{V}_\ell^{\,3}(z')}\,dz'$$

where

$$Q(z) = \int_0^z \frac{\varphi(z')}{\varphi_0}\left[1 - e^{-s\bar{\tau}(z')}\right]\frac{d\bar{V}}{dz'}(z')\,dz' \qquad 11.47$$

and all terms of $O(M^2)$ or higher, have been neglected. We conclude that under the present assumptions, $\frac{q(z)}{\varphi_0}$ is of $O(1)$ locally, if $\frac{d\bar{V}}{dz}(z)$ is of $O(1)$.

12. Solution by Iteration

In Section 10, we derived expressions for $\frac{\gamma\nu(z)}{\varphi_0}$, $\frac{\varphi(z)}{\varphi_0}$, and $\frac{\gamma\epsilon(z)}{\varphi_0}$, which are given respectively by Eqs. 10.27, 10.28 and 10.32. We must therefore examine the order of magnitude of the following integrals which appear in the aforementioned equations:

$$\int_0^z \gamma E(z') \begin{matrix}\sinh\\ \cosh\end{matrix}\, s(z-z')\,dz'$$
$$\int_0^z \gamma F(z') \begin{matrix}\sinh\\ \cosh\end{matrix}\, s(z-z')\,dz' \qquad 12.1$$

where $E(z')$ and $F(z')$ are defined through Eqs. 10.11, 10.13 and 10.17. Restricting our attention to the case $S = O(1)$, an analysis of the contributing terms shows that for the purposes of evaluating these integrals to within terms of $O(M)$, we may take:

$$\gamma E(z') = (3-\gamma)\bar{V}(z')\frac{\gamma \mathcal{V}(z')}{\varphi_0} + \frac{\gamma M(w)}{s} e^{-s\int_0^{z'}\frac{dz''}{\bar{V}_\ell(z'')}} \frac{d\bar{V}}{dz'}(z') \qquad 12.2$$

$$\begin{aligned}\gamma F(z') = {} & \gamma n Q(z') + \gamma \int_0^{z'} \left\{ H(w) - m\bar{\tau}(z'') s G(w) \right\} e^{-s\tau_t(z'')} \frac{d\bar{V}(z'')}{dz''} dz'' \\ & - \gamma \bar{V}_{\ell 0}^2 J(w) e^{-s\int_0^{z'}\frac{dz''}{V_{\ell z}(z'')}} \bar{V}_\ell(z') \frac{d\bar{V}(z')}{dz} \int_0^{z'} \frac{e^{-k\int_0^{z''}\frac{dz''}{\bar{V}_\ell(z'')}}}{\bar{V}_\ell^3(z'')} dz'' \\ & - (\gamma-1)\bar{V}(z')\frac{\varphi(z')}{\varphi_0} + \frac{\gamma k}{s}\bar{\rho}_\ell(z')\frac{\mathcal{V}(z')}{\varphi_0} - \gamma \bar{V}_\ell(z')\frac{1}{\varphi_0}\frac{dg}{dz'}(z')\end{aligned} \qquad 12.3$$

where

$$\frac{1}{\varphi_0}\frac{dg(z')}{dz'} = -\bar{V}_{\ell 0}^2 J(w) \int_0^{z'} \frac{e^{-k\int_0^{z''}\frac{dz'''}{\bar{V}_\ell(z''')}}}{\bar{V}_\ell^3(z'')} dz'' \frac{d}{dz'}\left\{ \bar{V}_\ell(z')\frac{d\bar{V}}{dz'} e^{-s\int_0^{z'}\frac{dz''}{\bar{V}_\ell(z'')}} \right\} \qquad 12.4$$

and $Q(z')$ has been defined in Eq. 11.47. We note that both $F(z')$ and $Y(z')$ which depend on $\frac{g(z')}{\varphi_0}$ are locally of O(1) because of the contribution of the term given by Eq. 11.45, but upon integration this term goes to O(M) since $\int_0^z \frac{d\bar{V}(z')}{dz'}dz'$ is O(M), and consequently $F(z')$ and $Y(z')$ must demonstrate the same behavior. Thus all terms which do not appear explicitly in Eqs. 12.2 and 12.3 yield contributions of O(M^2) or higher after integration, while those which are retained yield contributions of O(M).

Since $\bar{V}_{\ell 0}$ = O(M), the product $\bar{\rho}_{\ell 0}\bar{V}_{\ell 0}^2 J(w)$ is O(M^2), and Eqs. 10.27 and 10.28 reduce to:

$$\begin{aligned}\frac{\gamma \mathcal{V}(z)}{\varphi_0} = {} & \gamma Y(z) - \sinh sz + s\int_0^z \left[\gamma E(z')\cosh s(z-z')\right]dz' \\ & + s\int_0^z \left[\gamma F(z')\sinh s(z-z')\right]dz'\end{aligned} \qquad 12.5$$

$$\begin{aligned}\frac{\varphi(z)}{\varphi_0} = {} & -\gamma W(z) + \cosh sz - s\int_0^z \left[\gamma E(z')\sinh s(z-z')\right]dz' \\ & - s\int_0^z \left[\gamma F(z')\cosh s(z-z')\right]dz'\end{aligned} \qquad 12.6$$

where from Eqs. 10.11 and 10.13, we have correct to O(M):

$$\gamma Y(z) = \frac{\gamma g(z)}{\varphi_0} - \gamma \bar{V}(z)\frac{\varphi(z)}{\varphi_0} + O(M^2) = O(1) \qquad 12.7$$

$$\gamma W(z) = 2\bar{V}_z(z)\frac{\gamma \mathcal{V}(z)}{\varphi_0} + O(M^2) = O(M)$$

And now since $\sinh sz = i \sin wz$ and $\cosh sz = \cos wz$ we may rewrite Eqs. 12.5 and 12.6 in the form:

$$\frac{\gamma v}{\varphi_0}(z) = \frac{\gamma g(z)}{\varphi_0} - \sinh sz + O(M) = \frac{\gamma g(z)}{\varphi_0} - i\sin wz + O(M) \qquad 12.8$$

$$\frac{\varphi}{\varphi_0}(z) = \cosh sz + O(M) = \cos wz + O(M) \qquad 12.9$$

while examination of Eq. 10.32 shows that the entropy term is:

$$\frac{\gamma \epsilon(z)}{\varphi_0} \leq O(1) \qquad 12.10$$

This form of the equations suggests the following iteration procedure . First neglect the terms of O(M) with respect to terms of O(1), and then utilize the zeroth order solution to evaluate the higher order terms. The resulting zeroth order solution is:

$$\left(\frac{\gamma v(z)}{\varphi_0}\right)^{(0)} = \left(\frac{\gamma g(z)}{\varphi_0}\right)^{(0)} - i \sin wz$$

$$\left(\frac{\varphi(z)}{\varphi_0}\right)^{(0)} = \cos wz \qquad 12.11$$

correct to terms of O(1), where from Eq. 11.46 the term which is locally of O(1) is:

$$\left(\frac{\gamma g(z)}{\varphi_0}\right)^{(0)} = -\gamma \bar{V}_\ell(z)\frac{d\bar{V}}{dz}(z)\, J(w)\, \bar{V}_{\ell_0}^{\,2}\, e^{-s\int_0^z \frac{dz'}{\bar{V}_\ell(z')}} \int_0^z \frac{e^{-k\int_0^{z'} \frac{dz''}{\bar{V}_\ell(z'')}}}{\bar{V}_\ell^{\,3}(z')} \qquad 12.12$$

As we pointed out in Section 7, when the Mach number is identically zero, there is no combustion, $\frac{\gamma g(z)}{\varphi_0} = 0$, and there is no outflow, so that Eqs. 12.11 coincide precisely with the acoustic solution. That is, the phenomenon is reduced to the one-dimensional oscillation in a cylinder closed at both ends. At the chamber exit, $z = 1$, so that the corresponding

eigenvalues satisfy the equation:

$$\omega = m\pi \qquad (m = 0, 1, 2, 3 \ldots) \tag{12.13}$$

characteristic of organ-pipe oscillations. Furthermore, the oscillations must be neutral at these well-defined frequencies.

If combustion occurs, and the Mach number is small, but different from zero, terms of order M are added into Eqs. 12.8 and 12.9, but of greater significance is the fact that the boundary condition at $z = 1$, is no longer given by $\frac{\gamma \nu_e}{\varphi_o} = 0$, but rather by:

$$\frac{\gamma \nu_e}{\varphi_o} = \alpha_n \bar{V}_e \frac{\varphi_e}{\varphi_o} + \beta_n \bar{V}_e \frac{\gamma \epsilon_e}{\varphi_o} \tag{12.14}$$

This relation is the one-dimensional form of Eq. 7.7 derived by Crocco in Ref. 19, (see Appendix A). We note that α_n and β_n are complex functions of the frequency and nozzle geometry, and hence it follows that since we have changed the boundary condition, the values of ω for neutral oscillations are no longer given by Eq. 12.13, even if terms of O(M) are neglected. Leaving ω for the moment as an unknown eigenvalue to be determined later, we may now set down the equations for the perturbations which will be utilized in conjunction with Eq. 12.14. First we note that:

$$\begin{aligned} \gamma Y(z_e) &= \frac{\gamma g(z_e)}{\varphi_o} - \gamma \bar{V}_e \left(\frac{\varphi}{\varphi_o}\right)^{(0)} \\ &= \gamma n \int_0^1 \cos\omega z' \left[1 - e^{-s\bar{\tau}(z')}\right] \frac{d\bar{V}}{dz'}(z')\,dz' \\ &\quad + \gamma \int_0^1 \left\{ H(\omega) - m\bar{\tau}(z')\, s\, G(\omega) \right\} e^{-s\bar{\tau}_t(z')} \frac{d\bar{V}}{dz'}(z')\,dz' \\ &\quad - \gamma \bar{V}_e \cos\omega \end{aligned} \tag{12.15}$$

$$\gamma W(z_e) = 2\bar{V}_e \left(\frac{\gamma \nu_e}{\varphi_o}\right)^{(0)} = -2i\bar{V}_e \sin\omega$$

where these results are obtained from Eqs. 12.7 and 11.46, and the fact that combustion is complete at $z = z_e = 1$, so that $\left.\frac{d\bar{V}}{dz}\right)_{z_e} = 0.$

Evaluating the integrals at the chamber exit, we obtain from Eqs. 12.5, 12.6 and 10.32:

$$\left(\frac{\gamma \nu_e}{\varphi_0}\right)^{(1)} = \gamma Y^{(0)}(z_e) - i\sin w + iw\int_0^1 \left[\gamma E^{(0)}(z')\cos w(1-z')\right]dz' - w\int_0^1 \left[\gamma F^{(0)}(z')\sin w(1-z')\right]dz' \qquad 12.16$$

$$\left(\frac{\varphi_e}{\varphi_0}\right)^{(1)} = -\gamma W^{(0)}(z_e) + \cos w + w\int_0^1 \left[\gamma E^{(0)}(z')\sin w(1-z')\right]dz' - iw\int_0^1 \left[\gamma F^{(0)}(z')\cos w(1-z')\right]dz' \qquad 12.17$$

$$\left(\frac{\gamma \epsilon_e}{\varphi_0}\right)^{(1)} = (\gamma-1)\Bigg\{\bar{V}_e i\sin w - \frac{1}{\bar{V}_e}\int_0^1 e^{iw\int_1^{z'}\frac{dz''}{\bar{V}(z'')}}\cos wz'\frac{d\bar{V}}{dz'}(z')dz' + \frac{1}{\bar{V}_e}\int_0^1 e^{iw\int_1^{z'}\frac{dz''}{\bar{V}(z'')}}\bar{V}(z')\, w\sin wz'dz'\Bigg\} + \frac{\gamma M(w)}{\bar{V}_e}\int_0^1 e^{-iw\int_0^{z'}\frac{dz''}{\bar{V}_\ell(z'')}} e^{iw\int_1^{z'}\frac{dz''}{\bar{V}(z'')}}\frac{d\bar{V}}{dz'}(z')dz' \qquad 12.18$$

and it is noted that $\left(\frac{\gamma \nu(z')}{\varphi_0}\right)^{(0)}$ and $\left(\frac{\varphi(z')}{\varphi_0}\right)^{(0)}$ are given by Eq. 12.11 and are to be utilized in evaluating the integrals of $\gamma E^{(0)}(z')$ and $\gamma F^{(0)}(z')$ wherever applicable.

13. Discussion of the Characteristic Equation

We may now state the stability problem as follows: for a given injector, chamber geometry, distribution of combustion and exhaust nozzle, will an arbitrary perturbation of the steady state conditions be amplified or damped?

Note that comparison with Section 8 shows that several new factors have been introduced into the mathematical formulation. That is, the steady state distribution of combustion is still determined by $\bar{V}(z)$, but now in considering the unsteady effects, due regard must be taken of the injection system geometry and characteristic time a , and also the effect of the second interaction index m . The revised statement of the stability problem therefore becomes: for given a , $\bar{V}(z)$, $\bar{\tau}(z)$, $\bar{\tau}_t(z)$, α_n , β_n , m and n , will arbitrary perturbations be amplified or damped? As before, we will determine the stability boundary, i.e., we will solve for the neutral condition $\Lambda = 0$, $s = i\omega$.

If α , $\bar{V}(z)$, $\bar{\tau}_t(z)$, α_n and β_n are held fixed, these neutral conditions are possible only when a certain relation involving $\bar{\tau}(z)$ and the indices m and n is satisfied, and they will take place with a well determined frequency ω . Eq. 12.14 represents the functional relationship between the four quantities, and since it is a complex equation, it represents two real equations relating the sensitive time lag $\bar{\tau}(z)$ and neutral frequency ω to the indices m and n . For simplicity, we will suppose that the sensitive time lag is the same for all propellent elements, where $\bar{\delta}$ represents the critical value of the sensitive time lag. Thus for given values of m and n , it is possible to determine the values of the time lag $\bar{\delta}$ and the frequency ω for which neutral conditions can be obtained. Eq. 12.14 is the characteristic equation for the set of eigenvalues $\bar{\delta}$ and ω . However, as indicated in Section 8, the most convenient procedure is to prescribe the value of ω, and then to look for the eigenvalues m, n , and $\bar{\delta}$ compatible with neutral oscillations for that value of ω .

Now substituting Eqs. 12.16, 12.17 and 12.18 into Eq. 12.14, we obtain, (see Appendix C):

$$h_1 m \bar{\delta} = h_2 n (1 - e^{-i\omega\bar{\delta}}) + h_3 \qquad 13.1$$

where h_1, h_2 and h_3 are complex numbers which are a unique function of the frequency once the rocket chamber geometry, the injection and exhaust systems and the distribution of combustion have been specified. Therefore Eq. 13.1 is the final form of the characteristic equation of the rocket chamber and may be used to investigate the stability of a given rocket system. Note that this equation is somewhat more complicated than Eq. 8.2.

Separating the real and imaginary parts, Eq. 13.1 becomes:

$$m\bar{\delta} h_{1Re} = n\left[h_{2Re}(1-\cos\omega\bar{\delta}) - h_{2Im}\sin\omega\bar{\delta}\right] + h_{3Re} \qquad 13.2$$
$$m\bar{\delta} h_{1Im} = n\left[h_{2Im}(1-\cos\omega\bar{\delta}) + h_{2Re}\sin\omega\bar{\delta}\right] + h_{3Im}$$

Essentially then, we have two simultaneous equations of the form:

$$f_1(m, n, \bar{\delta}, \omega) = 0 \qquad 13.3$$
$$f_2(m, n, \bar{\delta}, \omega) = 0$$

Let us first eliminate m, and then we obtain the single equation:

$$n = \frac{b_1}{b_2(1-\cos\omega\bar{\delta}) - b_3\sin\omega\bar{\delta}} \qquad 13.4$$

where b_1, b_2 and b_3 are real numbers and are constant for a given value of the frequency,

$$b_1 = h_{1Re}h_{3Im} - h_{1Im}h_{3Re} \qquad 13.5$$
$$b_2 = h_{1Im}h_{2Re} - h_{1Re}h_{2Im}$$
$$b_3 = h_{1Im}h_{2Im} + h_{1Re}h_{2Re}$$

We can investigate the behavior of n in the $\omega\bar{\delta}$ plane. It is immediately noted that n must be a periodic function of $\omega\bar{\delta}$. By setting the denominator of Eq. 13.4 equal to zero, we may determine the values of $\omega\bar{\delta}$ for which n becomes infinite. We have:

$$b_2(1-\cos\omega\bar{\delta}) = b_3\sin\omega\bar{\delta} \qquad 13.6$$

Since η has a periodicity of 2π in the $\omega\bar{\delta}$ plane, we may confine our attention to the region $0 \leqq \omega\bar{\delta} \leqq 2\pi$. Solution of Eq. 13.6 yields:

$$\omega\bar{\delta} = 2K\pi \qquad (K = 0, 1 \cdots\cdots) \qquad 13.6$$

as well as:

$$\omega\bar{\delta} = B \qquad 13.7$$

where

$$\begin{cases} \cos B = \dfrac{b_2^2 - b_3^2}{b_2^2 + b_3^2} \\ \sin B = \dfrac{2b_2 b_3}{b_2^2 + b_3^2} \end{cases} \qquad 13.8$$

Since $1-\cos\omega\bar{\delta}$ approaches zero faster than $\sin\omega\bar{\delta}$, η will tend to infinity with the sign of $-b_1/b_3$ as $\omega\bar{\delta}$ approaches zero from the right, and η will tend to infinity with opposite sign as $\omega\bar{\delta}$ approaches 2π from the left. Hence $\omega\bar{\delta} = 0$, and $\omega\bar{\delta} = 2\pi$ constitute vertical asymptotes for η . The third vertical asymptote of η is given by Eqs. 13.7 and 13.8. Examination shows that η must approach infinity with the same sign as at the origin to the left of this asymptote and with opposite sign to the right. If η is a continuous function, it must take on a stationary value between each pair of asymptotic values. The two values of $\omega\bar{\delta}$ for which η is stationary may be determined by setting $\frac{d\eta}{d(\omega\bar{\delta})} = 0$. Accordingly,

$$\frac{b_1}{[b_2(1-\cos\omega\bar{\delta}) - b_3\sin\omega\bar{\delta}]^2}\left[b_2\sin\omega\bar{\delta} - b_3\cos\omega\bar{\delta}\right] = 0 \qquad 13.9$$

and hence assuming that b_1 is not identically zero, and that the decomina-tor does not vanish, we find that η is stationary when

$$b_2\sin\omega\bar{\delta} = b_3\cos\omega\bar{\delta} \qquad 13.10$$

There are two solutions to this equation.

$$\begin{cases} \cos\omega\bar{\delta}_1 = \dfrac{b_2}{\sqrt{b_2^2 + b_3^2}} \\ \sin\omega\bar{\delta}_1 = \dfrac{b_3}{\sqrt{b_2^2 + b_3^2}} \end{cases} \qquad 13.11$$

and $$\cos\omega\bar{\delta}_2 = \frac{-b_2}{\sqrt{b_2^2+b_3^2}} \qquad 13.12$$

$$\sin\omega\bar{\delta}_2 = \frac{-b_3}{\sqrt{b_2^2+b_3^2}}$$

it follows that $\omega\bar{\delta}_1$ and $\omega\bar{\delta}_2$ will lie in the first and third quadrants, or in the second and fourth, respectively. These results are pictured in Figure 13.1 and 13.2.

It is clear that if we wish to obtain the minimum value of n compatible with neutral oscillations at a given frequency ω, we may take either $\bar{\delta}_1$ or $\bar{\delta}_2$ as given by Eq. 13.11 or 13.12 and substitute back into Eq. 13.4. Only the positive values of n have physical significance.

With these values of n and $\bar{\delta}$, we can also substitute into Eq. 13.2 and obtain m. Thus m, n_{min} and $\bar{\delta}$ may be determined as a function of ω.

Let us examine the function $m(\omega\bar{\delta})$ more closely. We may proceed by eliminating n from Eqs. 13.2 and we obtain:

$$m\bar{\delta} = \frac{b_4(1-\cos\omega\bar{\delta}) - b_5\sin\omega\bar{\delta}}{b_2(1-\cos\omega\bar{\delta}) - b_3\sin\omega\bar{\delta}} \qquad 13.13$$

where

$$b_4 = h_{2Re}h_{3Im} - h_{2Im}h_{3Re}$$
$$b_5 = h_{2Re}h_{3Re} + h_{2Im}h_{3Im} \qquad 13.14$$

since the denominator of Eq. 13.13 is either finite or zero, $m\bar{\delta}(\omega\bar{\delta})$ can vanish only when the numerator vanishes. The numerator vanishes when $\omega\bar{\delta} = 0$ or 2π, but the denominator likewise vanishes there. However we have seen that $1-\cos\omega\bar{\delta}$ will vanish faster than $\sin\omega\bar{\delta}$, and hence

$$\lim_{\omega\bar{\delta}\to 0} m\bar{\delta} = \lim_{\omega\bar{\delta}\to 2\pi} m\bar{\delta} = \frac{b_5}{b_3} \qquad 13.15$$

But since $\bar{\delta} = 0$, at $\omega\bar{\delta} = 0$, we find $\mathcal{M}(0) = \infty$ with sign $\frac{b_5}{b_3}$, while at $\omega\bar{\delta} = 2\pi$, $\mathcal{M}(2\pi) = \frac{\omega}{2\pi}\frac{b_5}{b_3}$.

The numerator also vanishes when $\omega\bar{\delta}$ is given by:

$$\cos\omega\bar{\delta} = \frac{b_4^2 - b_5^2}{b_4^2 + b_5^2} \qquad \sin\omega\bar{\delta} = \frac{2b_4b_5}{b_4^2 + b_5^2} \tag{13.16}$$

Since the denominator of Eq. 13.13 doesn't vanish at this value of $\omega\bar{\delta}$, in general, and since $\bar{\delta}$ is finite, $\mathcal{M}$ vanishes at most only once at the value of $\omega\bar{\delta}$ given by Eq. 13.16.

IV. RESULTS AND CONCLUSIONS

14. Numerical Computations

Since we are dealing with a linearized analysis, all modes of oscillation, including the standing wave and traveling wave forms, can exist simultaneously and independently. However, all modes have a unique frequency for a fixed geometry and hence may be investigated independently of each other. The computation of the stability limits for a particular rocket motor proceeds directly once the chamber and exhaust nozzle geometry, injection system and the steady state distribution of combustion have been described, as discussed in Sections 8 and 13.

Thus, by way of illustration, a typical procedure would be as follows: specify

(a) the chamber geometry, length and diameter,

(b) the injection system,

(c) the subsonic portion of the exhaust nozzle, and

(d) the steady state distribution of gas velocity in the chamber.

In connection with the above items, it is noted that the drag coefficient of the droplets k, will depend on the droplet diameters and hence on the injection system, item b, as well as the viscosity of the gas. It is sufficient to specify only the subsonic portion of the DeLaval nozzle, item c, since the flared supersonic portion has no effect on the chamber oscillations. And finally, we note that since the description of the combustion process is obtained equally well by prescribing either the burning rate, or the velocity distribution, for convenience we may prescribe the latter taking care that the axial component of the steady state gas velocity is zero at the injector end, and has a vanishing spatial derivative at the exit end of the chamber. This last condition must be met if one wishes to satisfy the requirement that combustion is complete within the chamber.

Returning to Item b above, it is observed that many different types of injector response are included in the analysis of Appendix B, including for example:

(i) cavitating venturi injectors,

(ii) matched impedance injectors, and

(iii) mismatched impedance injectors.

An ideal cavitating venturi injection system would, of course, have zero response to chamber pressure oscillations, and hence result in the following simplifications:

$$G(\omega)_{c.v.} = H(\omega)_{c.v.} = J(\omega)_{c.v.} = M(\omega)_{c.v.} = 0 \qquad 14.1$$

If we define a matched impedance injector as one for which the mixture ratio alone does not oscillate (minimized entropy wave effects) when the chamber pressure oscillates, then the analysis of Appendix B shows that the two conditions which mu t be fulfilled are:

$$\begin{aligned} a_{ox} &= a_f \\ \left[\bar{V}_{l_o}^{2}\bar{\rho}_{l_o}\right]_{ox} &= \left[\bar{V}_{l_o}^{2}\bar{\rho}_{l_o}\right]_f \end{aligned} \qquad 14.2$$

If only the first condition in 14.2 is satisfied, then the oxidizer and fuel lines will have the same phase lag, but the injector will nevertheless produce an osicillating mixture ratio. This result follows because the twin conditions of equal phase lags and equal amplitudes must be met.

The relationships in Eqs. 14.2 may also be expressed as:

$$\begin{aligned} \bar{P}_{R_{ox}} &= \bar{P}_{R_f} \\ \left[\int_{z_R}^{0} \frac{dz}{A(z)}\right]_f \left[\int_{z_R}^{0} \frac{dz}{A(z)}\right]_{ox}^{-1} &= \Gamma \\ \frac{(A_o)_{ox}}{(A_o)_f} &= \Gamma\sqrt{\frac{(\rho_{l_o})_f}{(\rho_{l_o})_{ox}}} \end{aligned} \qquad 14.3$$

where as before, the subscripts ox and f refer to the oxidizer and fuel respectively. If all three conditions are satisfied, then:

$$G(\omega)_{M.I.} = M(\omega)_{M.I.} = 0 \qquad 14.4$$

to within terms of O(M) .

Thus, for cases I and II, Eq. C 21 shows that h_1 is identically zero and hence this means that when the mixture ratio is constant, the characteristic Equation 13. I can be reduced to the same form as Eq. 8.2 even when there is some other form of coupling between the chamber and the feedlines. Case III, of course, corresponds to the general case of a bipropellant injection system with arbitrary relative phase lag between the oxidizer and fuel line and arbitrary amplitude response for each line.

It is noted that for sufficiently high chamber frequencies, the functions $G(\omega), H(\omega), J(\omega)$ and $M(\omega)$ are negligible and h_1 again vanishes. Thus, to summarize, the characteristic equation of the chamber reduces to the form given by Eq. 8.2 when the mixture ratio is constant. This condition exists generally in a monopropellant motor, and in a bipropellant motor with an injector corresponding to cases I or II, and also as just observed, in a chamber in which only high frequency oscillations are present.

Once the propellant injection velocity is utilized in solving Eq. 4.11 for $\bar{V}_\ell(z)$, Eq. 5.7 may be rearranged to give:

$$\bar{\rho}_\ell(z) = \frac{\bar{V}_e - \bar{V}(z)}{\bar{V}_\ell(z)} \qquad 14.5$$

since $\bar{\rho}_{\ell i}\bar{V}_{\ell i} = \bar{\rho}_e\bar{V}_e$. It is then possible to evaluate all the integrals leading to the determination of the three complex quantities h_1 , h_2 and h_3 as a function of chamber frequency, and then to determine the stability limits of the chamber as given by the eigenvalues m, n and $\bar{\delta}$. In closure, it is remarked that the complexity of the integrals used in evaluating h_1, h_2 and h_3

precludes any but numerical evaluation of these functions.

Let us now consider the two rocket motors defined below:

Chamber No.	Type	r_c^*	L^*	z_e	r_{th}^*	$\bar{V}_{ze}$	ℓ^*_{sub}
1	Short	2.000"	2.000"	1.000	0.846"	0.1	0.728"
2	Short	2.000"	2.000"	1.000	1.150"	0.2	0.910"

where ℓ^*_{sub} is obtained from Eq. A 9 for K = 1.00 and where the steady state distribution of chamber velocity is given by :

$$0 \le z \le 0.2\, z_e \qquad \bar{V}_z = 0$$

$$0.2\, z_e \le z \le 0.5\, z_e \qquad \bar{V}_z = \frac{\bar{V}_{ze}}{0.3\, z_e}(z - 0.2\, z_e)$$

$$0.5\, z_e \le z \le z_e \qquad \bar{V}_z = \bar{V}_{ze}$$

and where k = 0.15 for $\bar{V}_{\ell_o}$ = 0.05 (See Figs. 4.2 and 14.1).

Since we have already outlined all of the fundamental concepts, solved for the eigenvalues in implicit form and briefly discussed the procedure to be followed in evaluating all of the component integrals, we need merely remark that we will investigate the first transverse mode for which S_{nh} = 1.84129, and then the values listed in Table III follow directly. Note that for all these cases, h_1 , is identically zero. These results are plotted in Figures 14.2 and 14.3.

If we examine the curve of n versus ω for chamber number 1 ($\bar{V}_{ze}$ = 0.1), we see that this curve does not exhibit a minimum; however, if the values for h_2 and h_3 are extrapolated* out to a value of ω = 1.50, then there is obtained:

ω	h_{2Re}	h_{2Im}	h_{3Re}	h_{3Im}	n	$\bar{\delta}$
1.60	0.148	0.0172	- 0.55	0.38	3.00	2.97
1.50	0.155	0.170	- 0.46	0.435	3.14	3.46

* A qualitatively correct, but not rigorous procedure.

and it is observed that n takes on its minimum value of approximately 2.99 in the vicinity of $\omega = 1.65$.

Examination of Figure 14.2 also shows that the curve of n for chamber number 2 exhibits a minimum of 2.44 in the vicinity of $\omega = 1.84$. Hence, a comparison of the results obtained for these two chambers seems to indicate that when the chamber exhaust velocity is larger, the stability limit is smaller, which means that the susceptibility of the chamber to combustion instability is then greater. (See Figure 14.3) We further note that the minimum n occurs at a different frequency for each of these chambers. That is, the chamber with the higher exhaust velocity (at the nozzle entrance) has a higher neutral frequency for minimum n.

A very interesting result is obtained in an investigation of transverse mode instability when due to some peculiar combination of events in the chamber, the complex quantity h_3 vanishes identically at a given chamber frequency. For such a situation:

$$h_2 n \left(1 - e^{-i\omega\bar{\delta}}\right) = 0 \qquad 14.6$$

so that either

$$n = 0 \qquad 14.7$$

or

$$\begin{cases} h_{2Re}\left(1 - \cos\omega\bar{\delta}\right) - h_{2Im} \sin\omega\bar{\delta} = 0 \\ h_{2Im}\left(1 - \cos\omega\bar{\delta}\right) + h_{2Re} \sin\omega\bar{\delta} = 0 \end{cases} \qquad 14.8$$

or both hold true simultaneously. Such a solution involves a certain amount of indeterminacy, however, some general conclusions may nevertheless be drawn.

If Equation 14.7 holds alone, then $\bar{\delta}$ is indeterminate. The stability limit is then given by the horizontal line $n = 0$ in the $n, \bar{\delta}$ plane. Since the n of any given propellant combination is generally finite, this would indicate that combustion instability can exist at that frequency for which h_3 vanishes, regardless of the value of the sensitive time lag.

Now, the simultaneous solution of Eqs. 14.8 yields:

$$\bar{\delta} = \frac{2K\pi}{\omega} \qquad (K = 0, 1, 2, 3, \ldots) \tag{14.9}$$

and in this instance, n is indeterminate. The stability limit may then be interpreted as the vertical line $\bar{\delta} = 0$, in the $n, \bar{\delta}$ plane. This solution implies that if a given propellant has a zero sensitive time lag, then combustion instability will exist at the frequency at which h_3 vanishes, regardless of the value of the interaction index n. Thus we conclude that the conditions that make h_3 tend to zero, promote instability.

If the behavior of h_2 and h_3, at frequencies above or below the ω for which h_3 vanishes, is such that n and $\bar{\delta}$ are both negative, then the chamber will be unstable for those frequencies. This follows from the definition of the stability boundary, since when the computed eigenvalues are both physically unobtainable, the amplification coefficient Λ is then finite.

15. Comparison with Experiment and Conclusions

In this section we shall endeavor to compare the theoretical results presented in the main body of this thesis with published experimental data. We have developed a theory for a complex physico-chemical phenomenon based on a hypothetical model of the combustion process. The justification for such an approach resides in the fact that not too much is known about the myriad factors which influence the behavior of the entire system. However, the validity of our approach can be assessed only in terms of a direct comparison

between the trends predicted by the theory and the results obtainable from experiment.

The primary consideration which prevents such a direct comparison is the fact that the theoretical treatment presented here is for the stability of a rocket system against disturbances of small amplitude (applicable to the onset of combustion instability), while the majority of published results deal with the fully developed instability characterized by non-linear effects including shock waves and viscous damping.

Because this situation was recognized by Crocco some time ago, an experimental program, supported by the Bureau of Aeronautics, Department of the Navy, has been under way at the James Forrestal Research Center, Princeton University, for the specific purpose of studying the development of combustion instability in liquid propellant motors. A comprehensive treatment of the experimental findings is given in the recent work of Matthews (Ref. 23). Since the major part of Matthews' work deals with the experimental determination of the combustion time lag in motors operating at essentially constant mixture ratio, with a modulated propellant injection system which produces a low frequency oscillation in injection velocity and propellant flow rate, we cannot make any direct comparison with his work. Additional experimental effort along the lines of the theory developed here is clearly required, before such a direct comparison is feasible.

In any case, we must look elsewhere for a verification of the analysis. The accurate determination of quantities oscillating at high frequencies is always a difficult undertaking, however, aside from flow visualization, the quantity which is perhaps most easily determined is the frequency of chamber pressure oscillations. Thus, our theoretical results may be compared with experiment with regard to two separate measurements or observations, the gas motion, and the chamber pressure frequency during

unsteady motor operation.

Let us begin with the motion of a gas particle. It is of some interest to first sketch the modes of oscillation at an axial station for the values of S_{nh} given in Eq. 5.25. This may be accomplished by noting that at an axial station, the perturbations in gas velocity and pressure may be written:

$$
\begin{aligned}
V_z' &\sim J_n(s_n h r)\cos n\theta\, e^{i\omega t} \\
V_r' &\sim \frac{dJ_n}{dr}(s_n h r)\cos n\theta\, e^{i\omega t} \\
V_\theta' &\sim \frac{J_n(s_n h r)}{r}\, n \sin n\theta\, e^{i\omega t} \\
p' &\sim J_n(s_n h r)\cos n\theta\, e^{i\omega t}
\end{aligned}
\qquad 15.1
$$

The sketches shown in Fig. 15.1 then represent the isobars and the instantaneous directions of the gas particle motion in a tangential-radial plane, at a given station z at time t. In order to consider what transpires as the time increases, we observe that the result depends on whether we have standing waves or traveling waves (or both) in the chamber. Standing waves may be identified by the presence of stationary nodes or nodal diameters, indicating that two trains of waves of equal amplitude and frequency, but out of phase by 180°, have traveled past each other continuously. Such a situation is possible only by the process of wave reflection at the boundaries of the rocket chamber.

The standing wave form is easily obtained by noting that increasing time progressively reverses the pattern of particle motion and changes the algebraic sign of the excess pressure, so that when $\omega t = \pi n$, the motion is completely reversed and a pressure deficiency exists where there was an

excess, and vice-versa. At $\omega t = 2\pi n$, the original picture is restored.

In the traveling wave form, or spinning form, the patterns shown in Fig. 15.1 can rotate continuously, since Eqs. 15.1 can also be written in the form:

$$
\begin{aligned}
V_z' &\sim J_n(s_n h r)\, e^{i(\omega t + n\theta)} \\
V_r' &\sim \frac{dJ_n}{dr}(s_n h r)\, e^{i(\omega t + n\theta)} \\
V_\theta' &\sim \frac{J_n(s_n h r)}{r}\, e^{i(\omega t + n\theta)} \\
p' &\sim J_n(s_n h r)\, e^{i(\omega t + n\theta)}
\end{aligned}
\qquad 15.2
$$

and it is clear that the time for a complete revolution is given by $\Delta t = \frac{2\pi n}{\omega}$. If we restrict our attention to the first tangential mode, and note that the particle velocity is superposed on a mean gas motion $\bar{V}_z$, the resultant particle motion may be sketched as in Fig. 15.2, in which the gas propagates axially as it spins, so that the motion of an individual gas particle is somewhat like a corkscrew. The number of complete revolutions that an individual gas particle makes in the chamber can be determined exactly by considering the Lagrangian derivative of the particle motion, however, an approximate value is given by:

$$N = \frac{\omega L}{2\pi n \bar{V}_{z\,avg.}} \qquad 15.3$$

Note that if there were no mean motion, as in a cylinder with closed ends, all the gas in the chamber would either "slosh" or "spin" simultaneously at every axial station. It is also observed that the linearized analysis permits

the simultaneous existence of any combination of standing and traveling waves with arbitrary direction of rotation.

In the experimental investigation reported in Ref. 15, probe microphones were utilized to determine the frequency and the amplitude and phase of the pressure oscillations. On the basis of their measurements, the authors claimed that the first transverse mode could be detected in the standing wave (sloshing) form. On the other hand, streak photographs have been taken through transparent slit windows by other investigators (see for example Ref. 21), which indicated the presence of a rotating luminous zone which propagates on a helical path along the chamber length. In this particular case, the authors state that their experimentally determined frequency is approximately that of the first transverse mode.

These results give credance to the possibility of obtaining both standing and traveling wave forms during unsteady operation of the combustion chamber. Our analysis anticipates the results obtained by frequency measurements since we have already observed that the frequency of neutral oscillations ω as determined by an actual solution for the eigenvalues ω and $\bar{\delta}$, will have the same order of magnitude as the value of ω given in Eq. 7.6 (the exact solution of the wave equation in a cylindrical chamber with closed ends). It is further noted that since the experimental value of the frequency during unsteady operation is very nearly given by Eq. 7.6, some investigators have assumed that the gas dynamical behavior of the chamber can be adequately described by the classical wave equation. This assumption obviously oversimplifies the actual state of affairs.

In conclusion, we remark that since the characteristic equation of a general rocket system has now been solved implicitly, for two important mechanisms capable of producing linear combustion instability in the high and intermediate frequency ranges, this solution may now be used as a research

tool in which a broad investigation of the behavior of different rocket motors is made. In such a parametric investigation, the susceptibility of the rocket motor to transverse wave or entropy wave instability can be determined for different

1) distributions of combustion
2) chamber geometry
3) injection systems
4) exhaust nozzles,

In view of the complicated form of the solution, it is unlikely that the general behavior of a system will be established without a major effort along the lines of additional numerical computations. The two numerical cases treated here represent two of the more interesting results which have already been obtained in this program of investigation. Great difficulties were encountered during the course of performing the computations. Negative results (uninteresting values of the eigenvalues) were obtained in a number of cases which were therefore not included in this presentation. It is felt that the two cases which are included in this analysis are typical of the results which may be obtained in an analytical investigation of combustion instability.

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TABLE I

Computed Values of Three-Dimensional Nozzle Admittance Coefficients

The following are held constant $\gamma = 1.20000$

$K = 1.00000$

$S_{nh} = 1.84129$

a. $\bar{V}_{ze} = 0.10000$

ω_{fig}	$\lvert\mathcal{A}\rvert$	$\mathcal{A}$ Phase	$\mathcal{A}_{Re}$	$\mathcal{A}_{Im}$
0.7500	0.3275	2.5839 rad.	- 0.2779	0.1732
1.0000	0.2338	2.4506 rad.	- 0.1802	0.1490
1.2500	0.1318	2.4338 rad.	- 0.1001	0.0857

ω_{fig}	$\lvert\mathcal{B}\rvert$	$\mathcal{B}$ Phase	$\mathcal{B}_{Re}$	$\mathcal{B}_{Im}$
0.7500	0.3631	5.0850 rad.	- 0.1311	- 0.3381
1.0000	0.3079	4.6922 rad.	- 0.0062	- 0.3078
1.2500	0.2469	4.3085 rad.	- 0.0972	- 0.2270

ω_{fig}	$\lvert\mathcal{C}\rvert$	$\mathcal{C}$ Phase	$\mathcal{C}_{Re}$	$\mathcal{C}_{Im}$
0.7500	0.1663	3.2780 rad.	- 0.1650	- 0.0229
1.0000	0.1025	2.8142 rad.	- 0.0971	0.0329
1.2500	0.0626	2.3590 rad.	- 0.0444	0.0442

$\gamma = 1.20000$

$K = 1.00000$

b. $\bar{V}_{ze} = 0.20000$ $S_{nh} = 1.84129$

ω_{fig}	$\lvert A \rvert$	A Phase	A Re	A Im
0.7500	0.4273	2.8020 rad.	− 0.4025	0.1434
1.0000	0.3425	2.7630 rad.	− 0.3185	0.1264
1.2500	0.2565	2.8409 rad.	− 0.2450	0.0759

ω_{fig}	$\lvert B \rvert$	B Phase	B Re	B Im
0.7500	0.5098	5.4100 rad.	0.3291	− 0.3896
1.0000	0.4502	5.1420 rad.	0.1878	− 0.4105
1.2500	0.3874	4.8881 rad.	0.0677	− 0.3814

ω_{fig}	$\lvert C \rvert$	C Phase	C Re	C Im
0.7500	0.3245	3.5900 rad.	− 0.2920	− 0.1415
1.0000	0.2135	3.2450 rad.	− 0.2120	− 0.0220
1.2500	0.1438	2.9246 rad.	− 0.1405	0.0309

TABLE II

Computed Values of One-Dimensional Nozzle Admittance Coefficients

The following are held constant: $\gamma = 1.2000$

$K = 1.0000$

$S_{nh} = 0$

a. $\bar{V}_e = 0.10000$

ω_{fig}	$\lvert\alpha_n\rvert$	α_n Phase	α_n Re	α_n Im
0	0.1000	0 rad.	0.1000	0
0.0500	0.1509	0.8315 rad.	0.1017	0.1115
0.1000	0.2473	1.1235 rad.	0.1069	0.2230
0.1500	0.3539	1.2453 rad.	0.1155	0.3345
0.2000	0.4639	1.2930 rad.	0.1275	0.4460
0.2500	0.5715	1.3203 rad.	0.1417	0.5537
0.5000	1.1175	1.3339 rad.	0.2623	1.0863
0.7500	1.6399	1.2932 rad.	0.4496	1.5771
1.0000	2.1361	1.2430 rad.	0.6877	2.0224
1.5000	3.0121	1.1422 rad.	1.2520	2.7395
2.5000	4.3898	0.9789 rad.	2.4520	3.6420
3.0000	4.9315	0.9147 rad.	3.0100	3.9080

ω_{fig}	$\lvert\beta_n\rvert$	β_n Phase	β_n Re	β_n Im
0	0.5000	6.2832 rad.	0.5000	0
0.0500	-	-	-	-
0.1000	-	-	-	-
0.1500	-	-	-	-
0.2000	-	-	-	-
0.2500	0.4795	5.7143 rad.	0.4040	- 0.2583
0.5000	0.4260*	5.1480 rad.*	0.1800*	- 0.3860*
0.7500	0.3429	4.6029 rad.	- 0.0374	- 0.3409
1.0000	0.2329	4.0764 rad.	- 0.1379	- 0.1870
1.5000	0.1054	3.0257 rad.	- 0.1047	- 0.0122
2.0000	0.0255	1.6873 rad.	0.0031	- 0.0253

*Interpolated

b. $\bar{V}_e = 0.2000$

ωfig	$\lvert\alpha_n\rvert$	α_n Phase	$\lvert\beta_n\rvert$	β_n Phase
0	0.1000	0 rad.	0.5000	6.2832 rad.
0.2500	0.4153	1.2947 rad.	0.4831	5.8766 rad.
0.5000	0.8012	1.2662 rad.	0.4281	5.4802 rad.
0.7500	1.1684	1.2212 rad.	0.3678	5.1052 rad.
1.0000	1.5110	1.1618 rad.	0.2909	4.7623 rad.
1.5000	2.0942	1.0403 rad.	0.1506	4.2469 rad.
2.0000	-	-	0.0624	4.1263 rad.
2.5000	2.9469	0.8445 rad.	-	-

c. $\bar{V}_e = 0.3000$

ωfig	$\lvert\alpha_n\rvert$	α_n Phase	$\lvert\beta_n\rvert$	β_n Phase
0	0.1000	0 rad.	0.5000	6.2832 rad.
0.2500	-	-	0.4862	5.9644 rad.
0.5000	0.6273	1.2124 rad.	0.4396	5.6549 rad.
0.7500	0.9153	1.1695 rad.	0.3915	5.3742 rad.
1.0000	1.1760	1.1064 rad.	0.3264	5.1252 rad.
1.5000	1.6160	0.9700 rad.	0.2072	4.7700 rad.
2.0000	1.9550	0.8500 rad.	0.1268	4.6570 rad.

TABLE III

Results obtained for the two chambers defined in Section 14.

Chamber No. 1

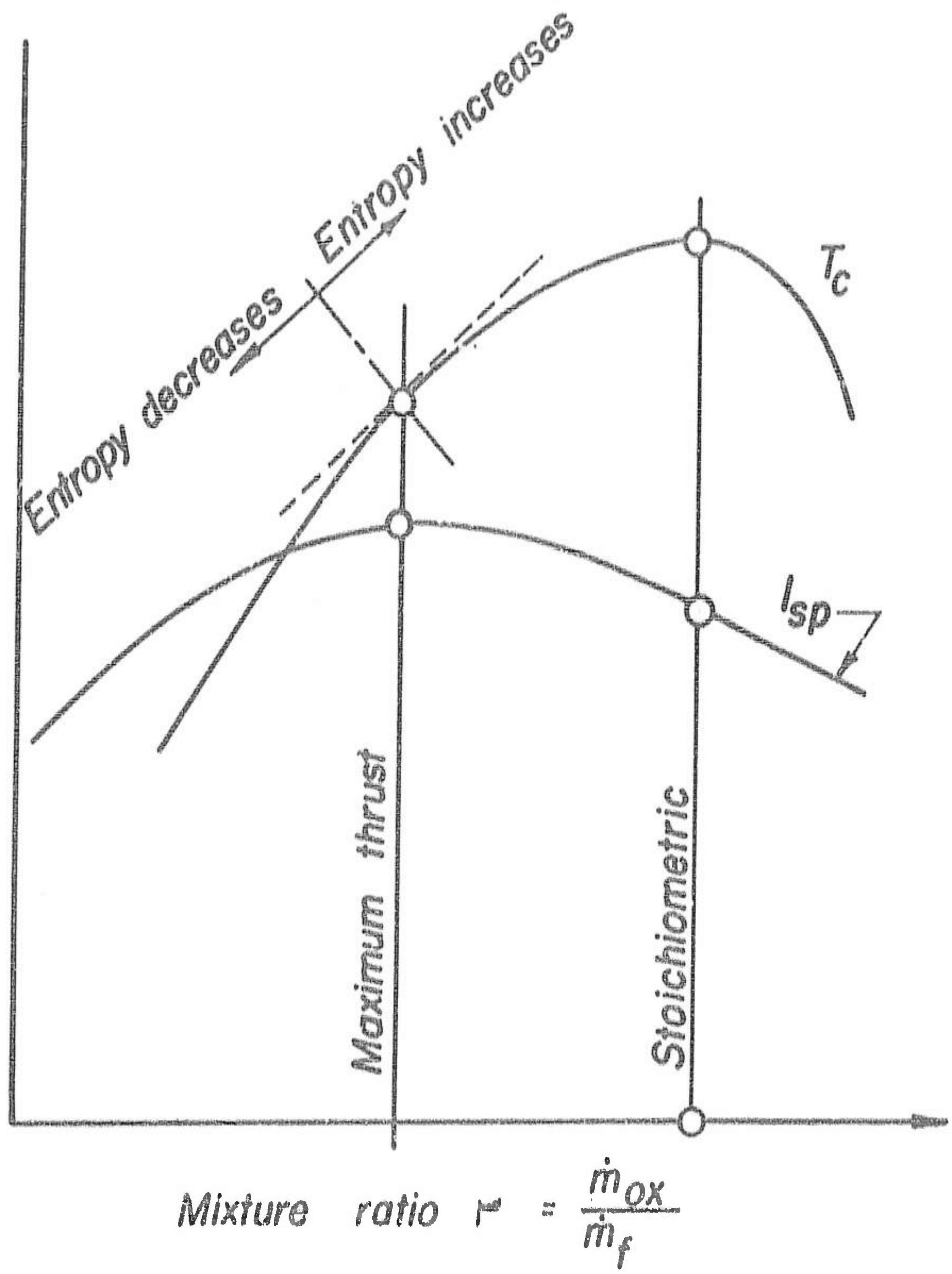
ω	h_{2Re}	h_{2Im}	h_{3Re}	h_{3Im}	n	$\bar{\delta}$
1.7000	0.1402	0.0173	- 0.6273	0.3226	3.0196	2.5509
1.7500	0.1361	0.0173	- 0.6666	0.2885	3.0770	2.4609
1.8000	0.1319	0.0168	- 0.7032	0.2557	3.1628	2.2733
1.8500	0.1278	0.0169	- 0.7286	0.2265	3.2606	2.1667
1.9000	0.1237	0.0164	- 0.7534	0.1962	3.3667	2.0604
1.9500	0.1196	0.0159	- 0.7727	0.1683	3.4839	1.9668

Chamber No. 2

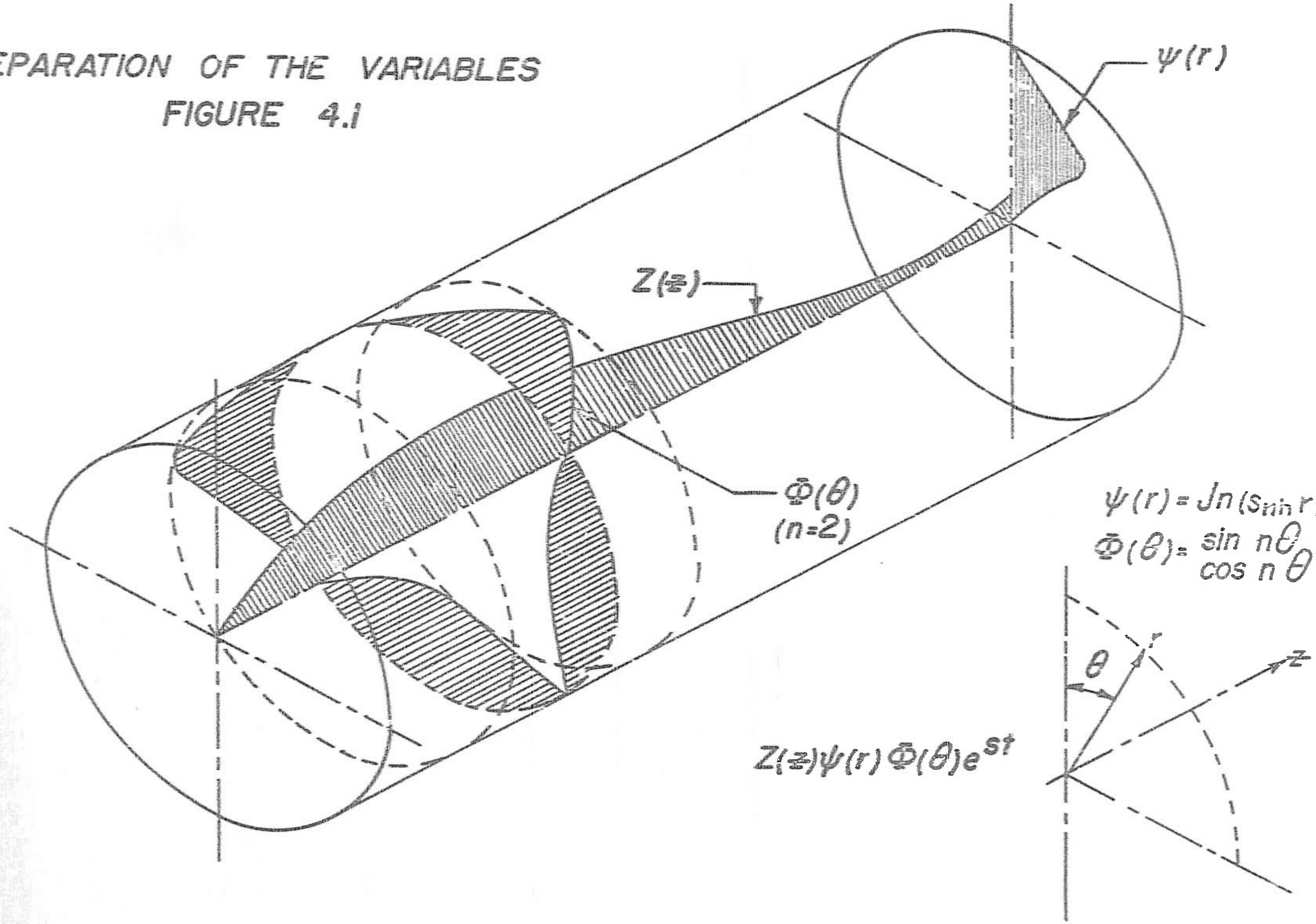
ω	h_{2Re}	h_{2Im}	h_{3Re}	h_{3Im}	n	$\bar{\delta}$
1.7000	0.2306	0.0502	- 1.1833	0.3870	2.4789	2.4281
1.7500	0.2717	0.0487	- 1.1750	0.3280	2.4540	2.3092
1.8000	0.2631	0.0467	- 1.1678	0.2769	2.4470	2.1993
1.8500	0.2542	0.0447	- 1.1554	0.2238	2.4405	2.0932
1.9000	0.2457	0.0436	- 1.1450	0.1857	2.4623	2.0077
1.9500	0.2369	0.0401	- 1.1316	0.1412	2.4774	1.9102

VARIATION OF COMBUSTION TEMPERATURE T_c AND SPECIFIC IMPULSE I_{sp} WITH MIXTURE RATIO μ

FIGURE 2.1



SEPARATION OF THE VARIABLES
FIGURE 4.1

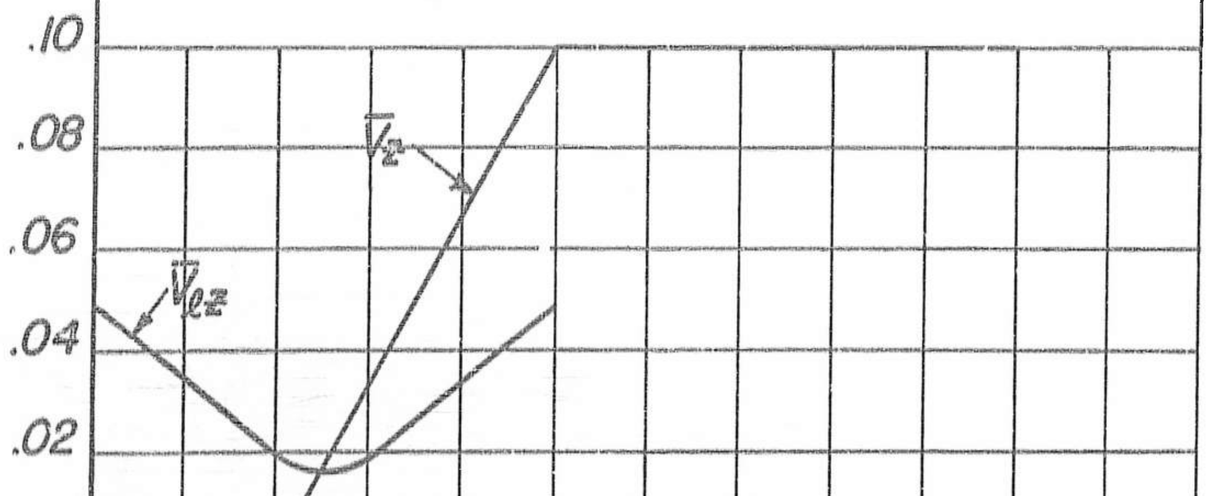


SOLUTION OF EQUATION 4.11)
FOR PROPELLANT DROPLET DYNAMICS

FIGURE 4.2

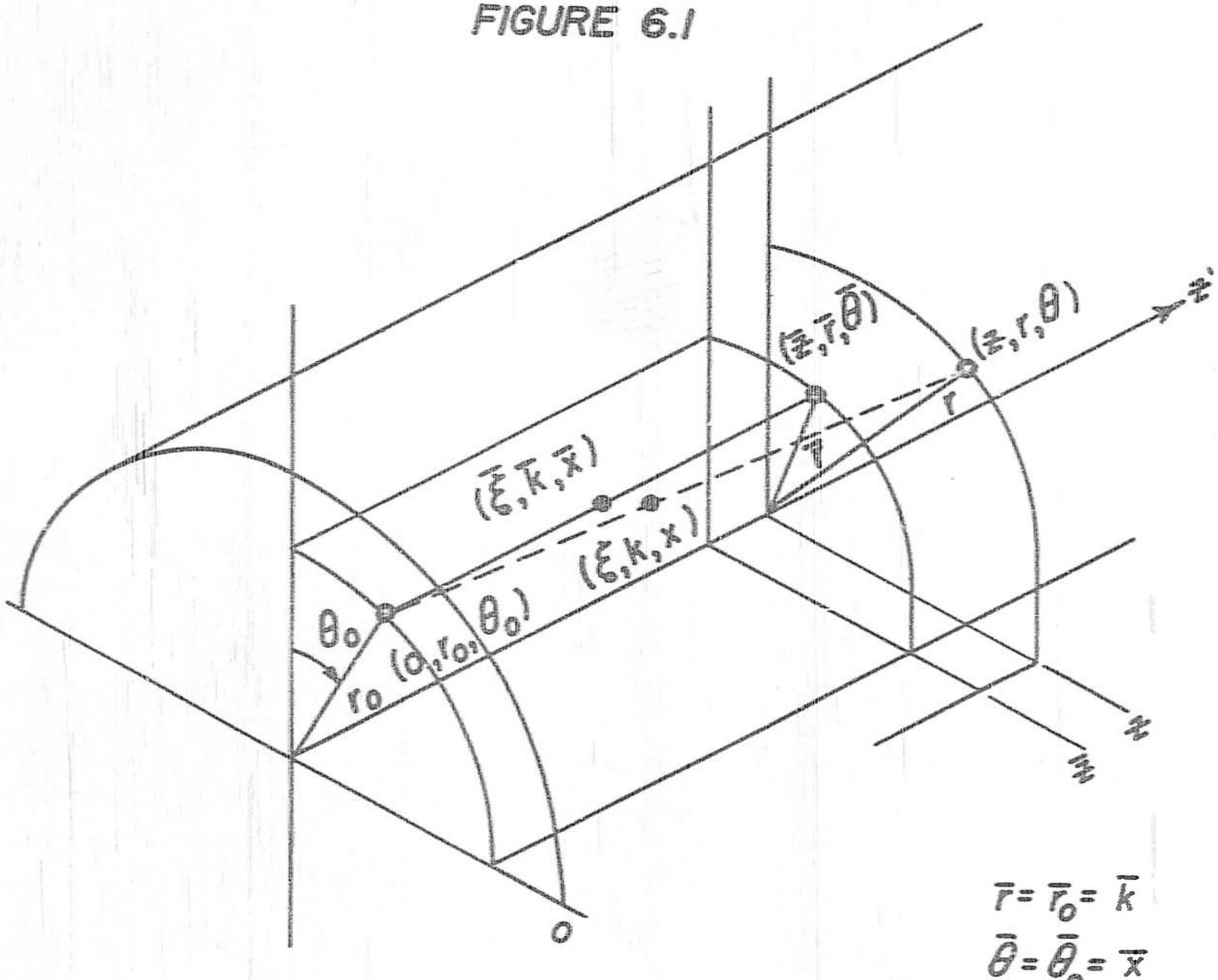
$$\bar{V}_{\ell z} \frac{d\bar{V}_{\ell z}}{dz} = k(\bar{V}_z - \bar{V}_{\ell z})$$

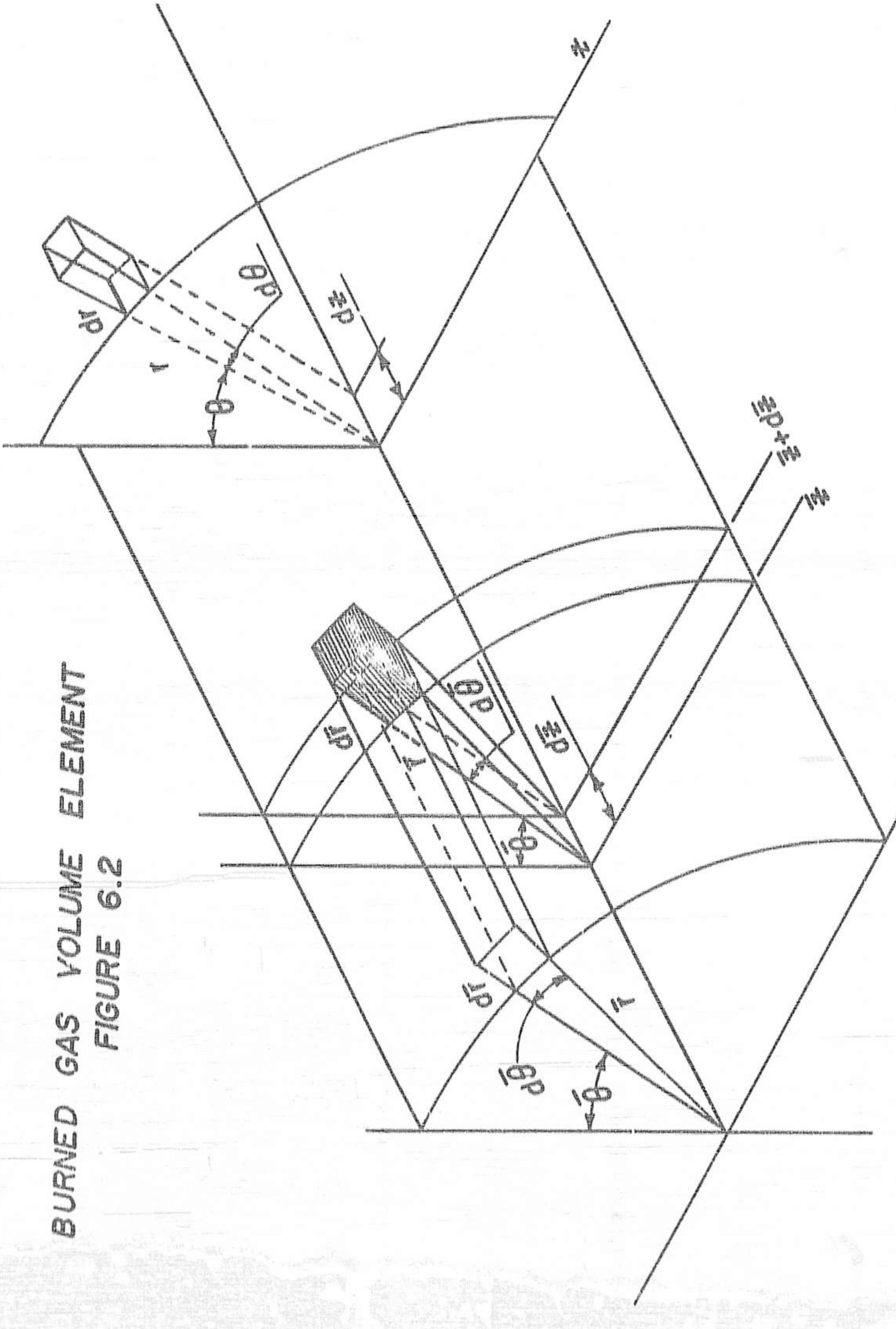
$k = 0.15$
$\bar{V}_{\ell o} = 0.05$
$\bar{V}_{ze} = 0.10$



SCHEMATIC OF THE SPACE LAG

FIGURE 6.1

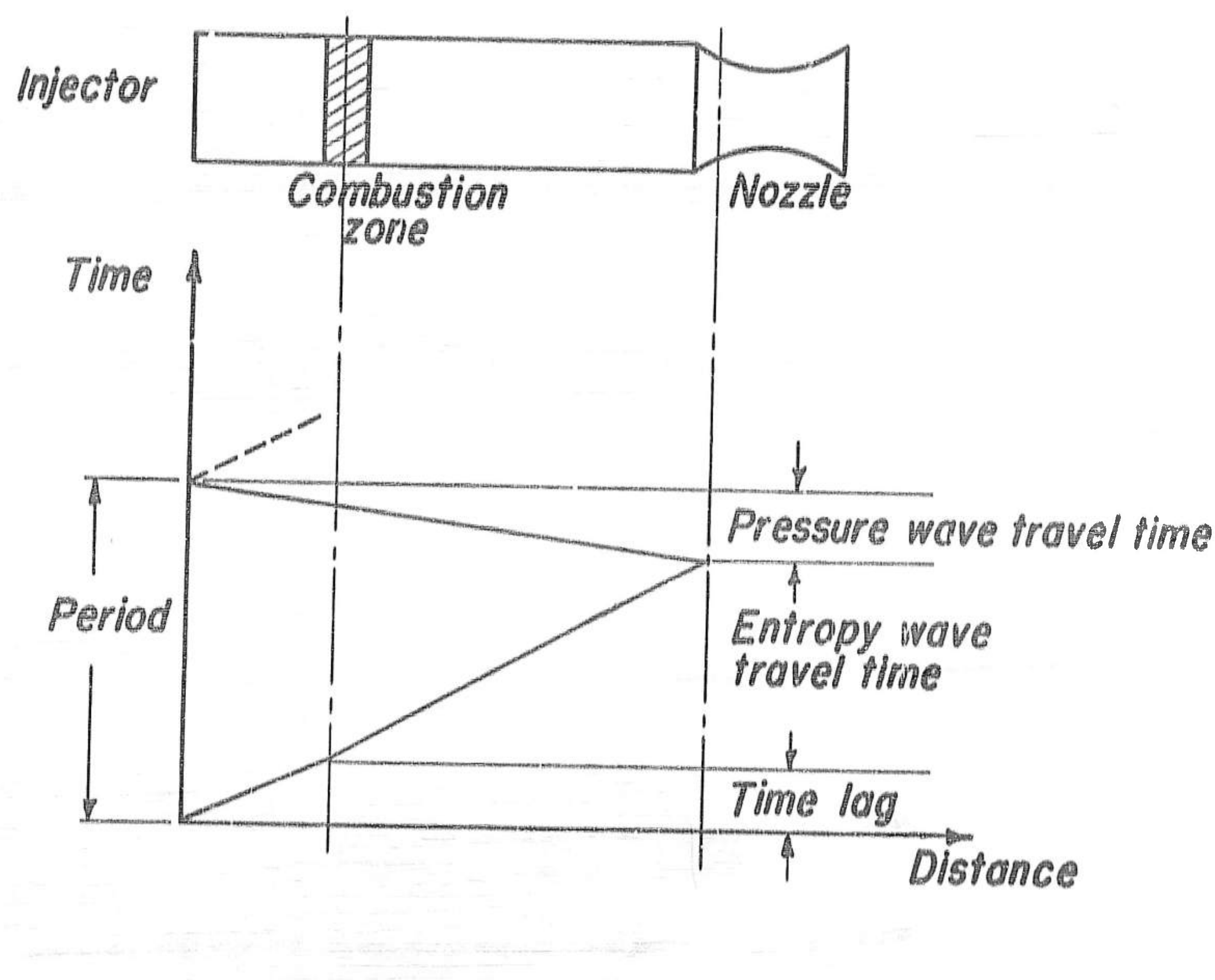




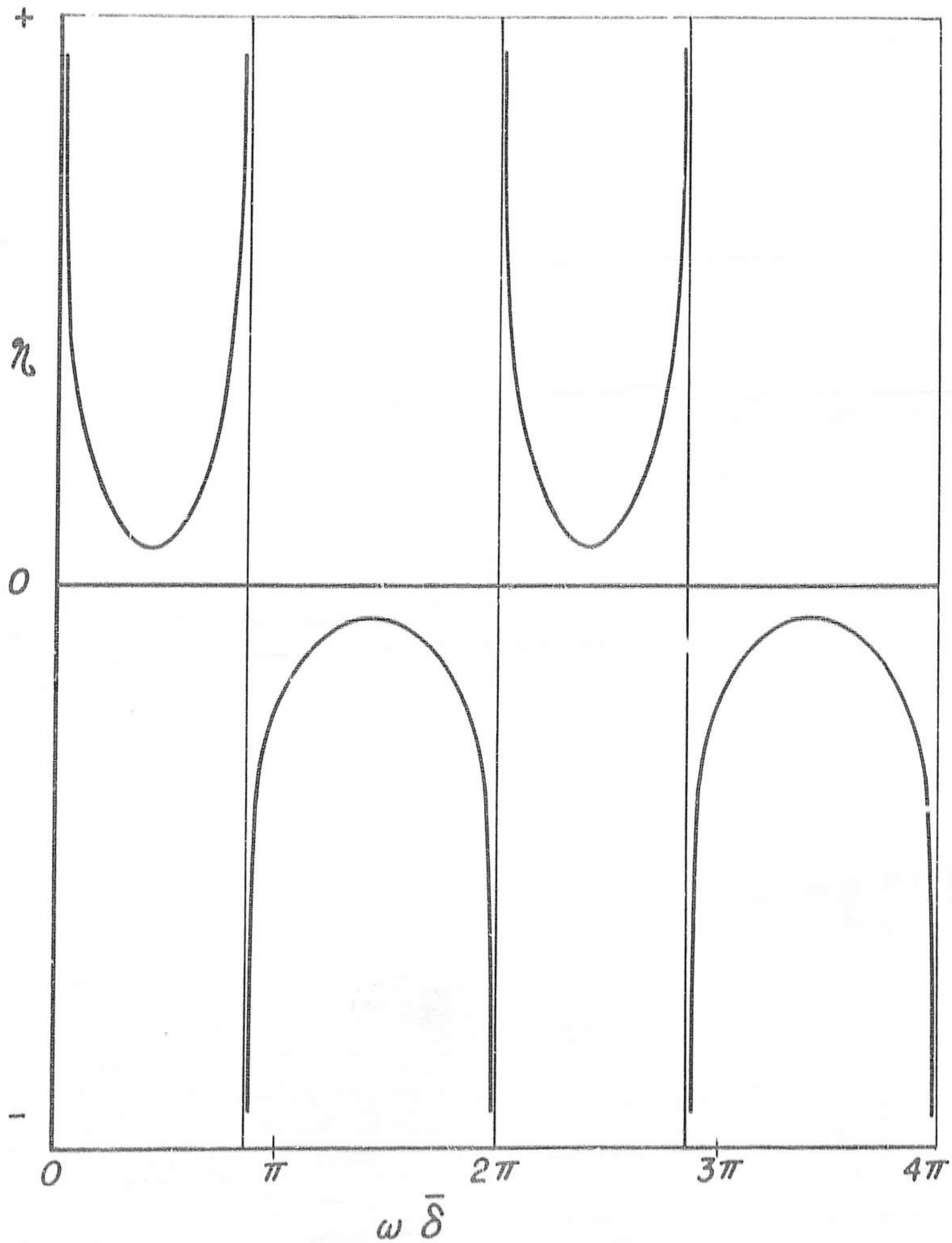
BURNED GAS VOLUME ELEMENT
FIGURE 6.2

SCHEMATIC REPRESENTATION OF THE MECHANISM FOR INTERMEDIATE FREQUENCY INSTABILITY

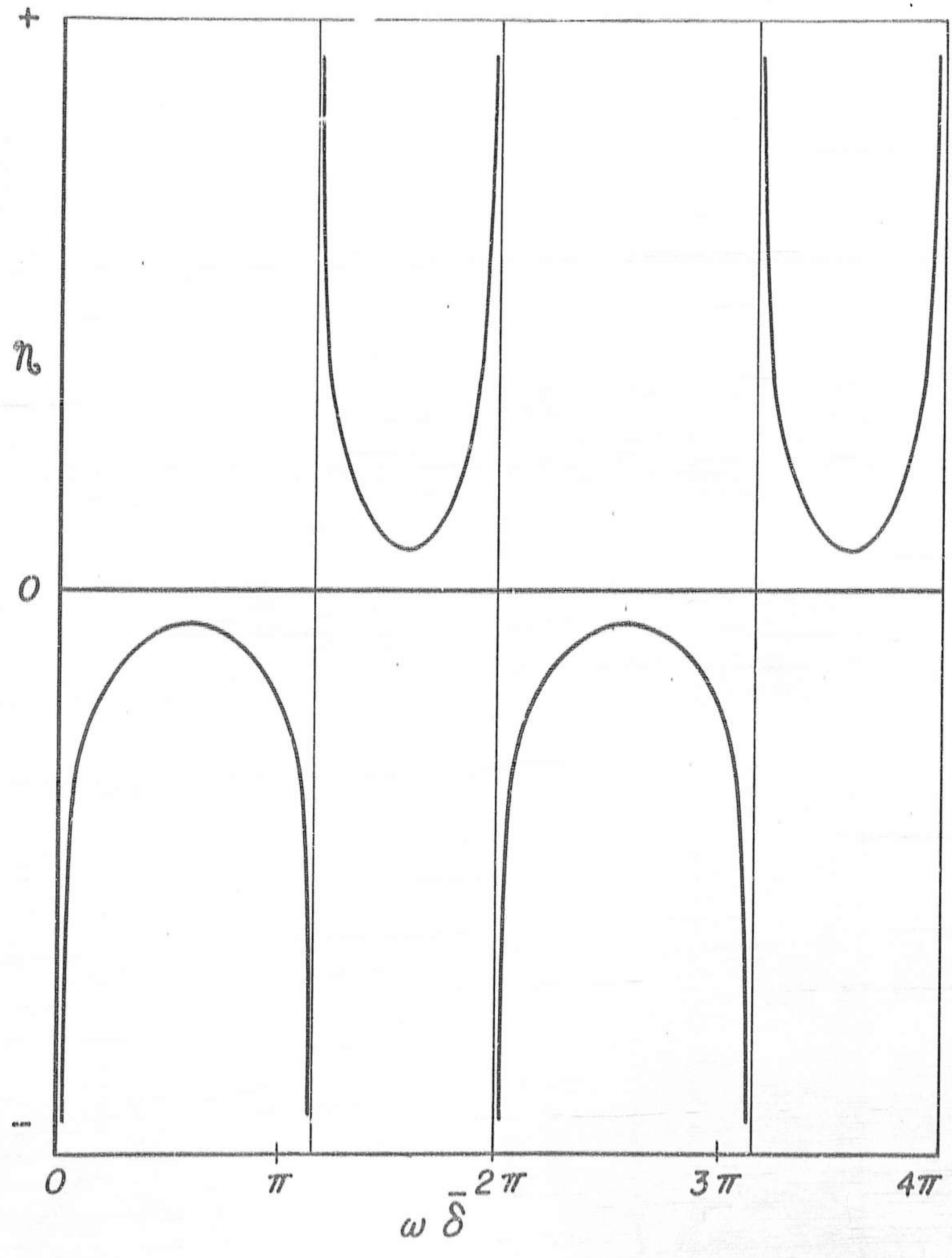
FIGURE 9.1



INTERACTION INDEX η VS $\omega\bar{\delta}$
(SOLUTION OF EQUATION 13.1)
FIGURE 13.1

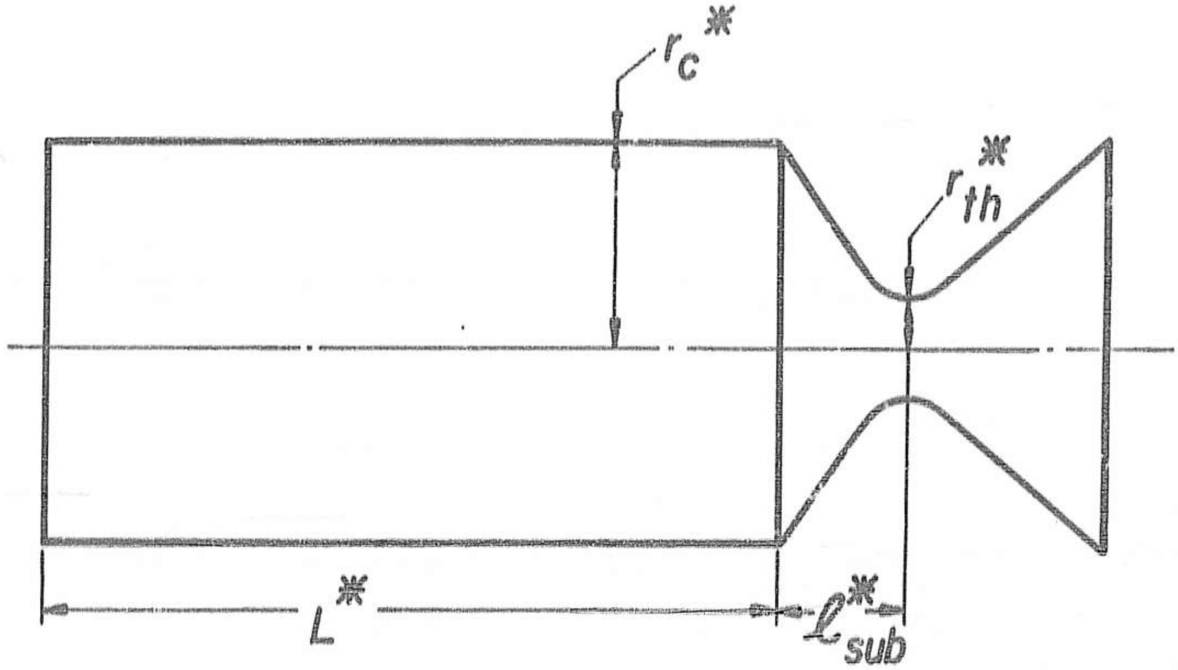


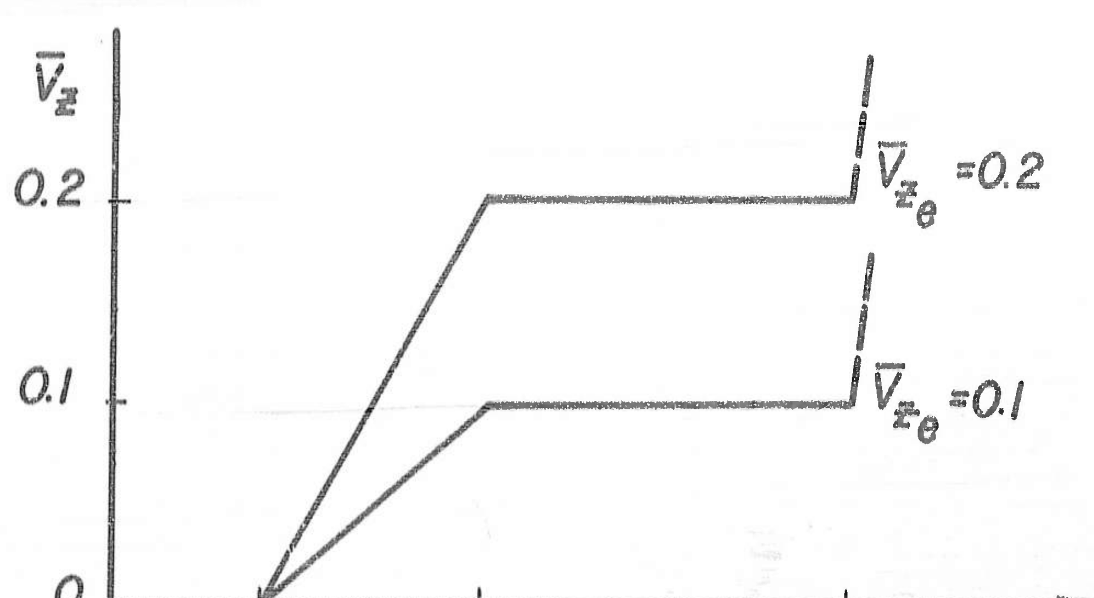
INTERACTION INDEX η VS $\omega\bar{\delta}$
(ALTERNATE SOLUTION OF EQUATION 13.1)
FIGURE 13.2

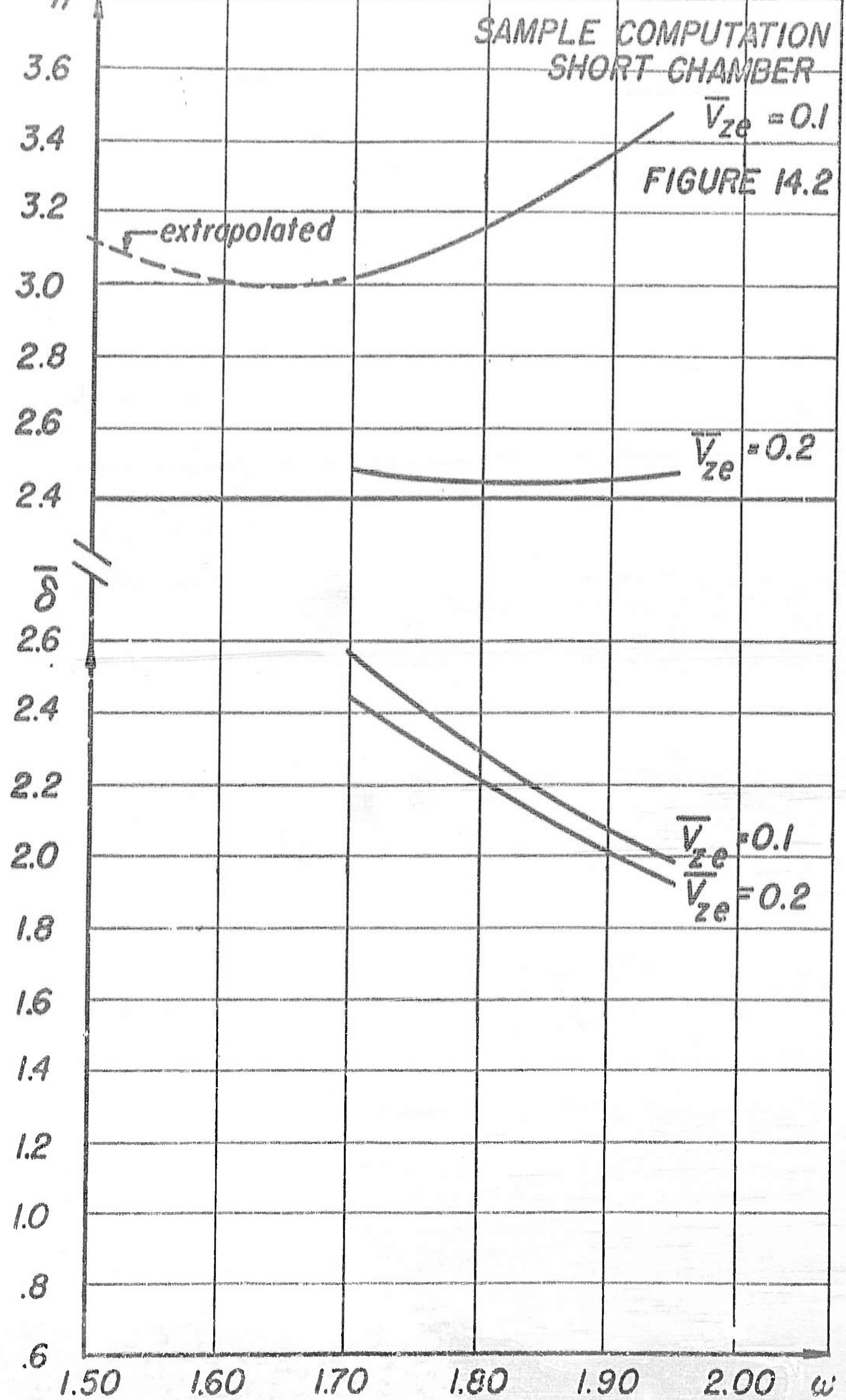


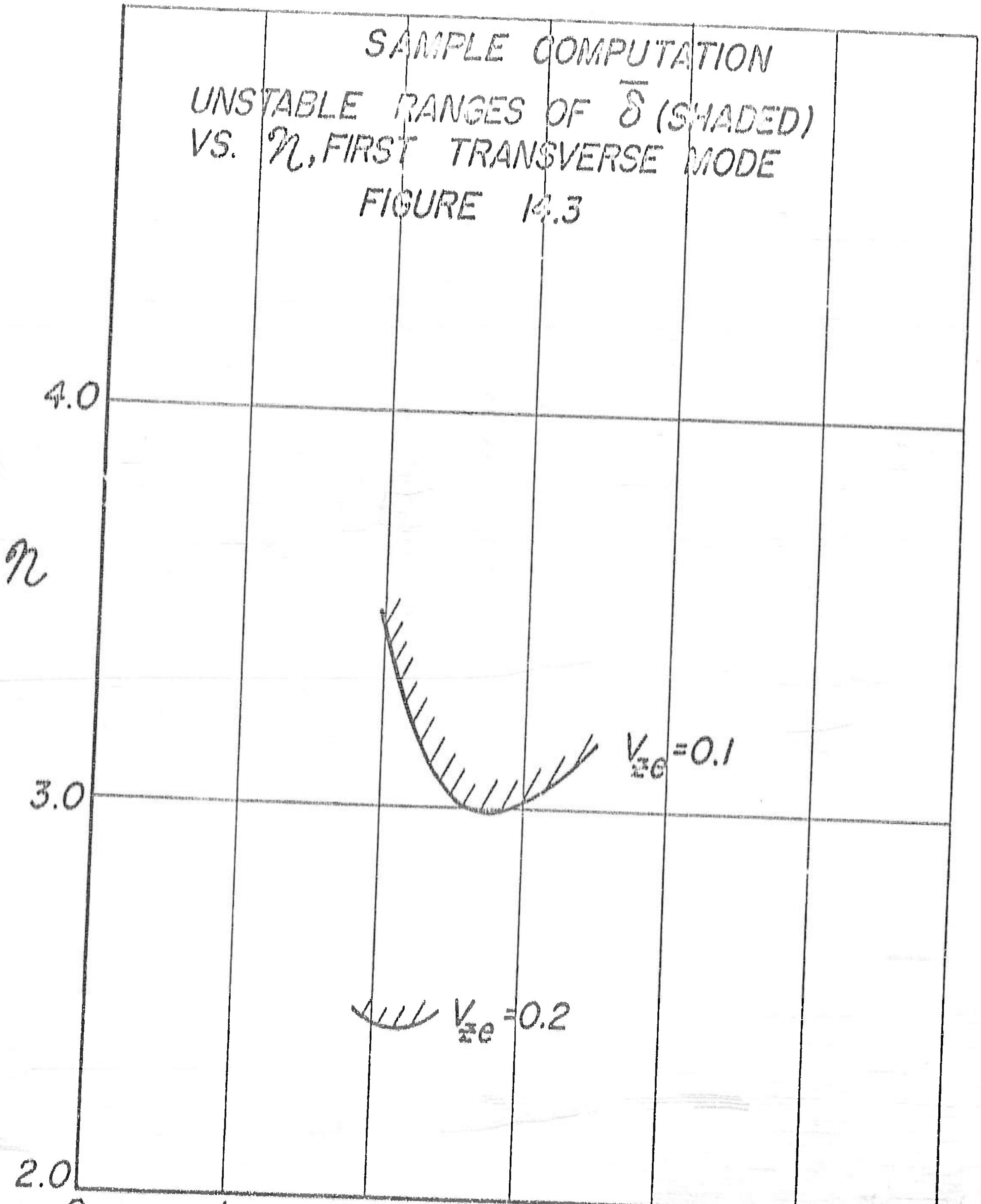
CHAMBER GEOMETRY AND STEADY STATE VELOCITY DISTRIBUTION

FIGURE 14.1

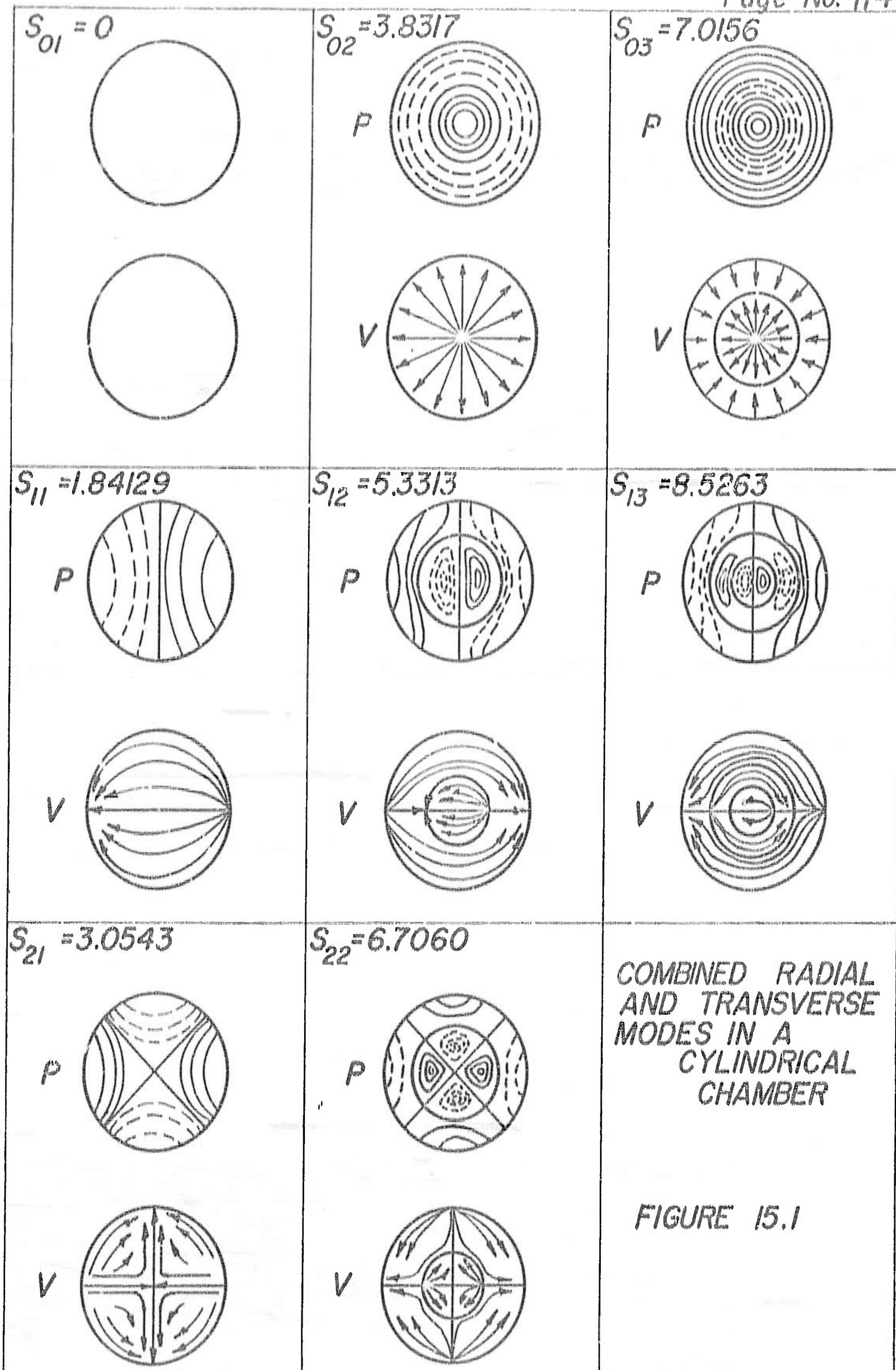





SAMPLE COMPUTATION
SHORT CHAMBER
FIGURE 14.2
η
$\bar{\delta}$
extrapolated
$\bar{V}_{ze} = 0.1$
$\bar{V}_{ze} = 0.2$
$\bar{V}_{ze} = 0.1$
$\bar{V}_{ze} = 0.2$
3.6
3.4
3.2
3.0
2.8
2.6
2.4
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
1.50
1.60
1.70
1.80
1.90
2.00
ω



SAMPLE COMPUTATION
UNSTABLE RANGES OF $\bar{\delta}$ (SHADED) VS. η, FIRST TRANSVERSE MODE
FIGURE 14.3



COMBINED RADIAL AND TRANSVERSE MODES IN A CYLINDRICAL CHAMBER

FIGURE 15.1

PROPAGATION OF FIRST TRANSVERSE MODE

SCHEMATIC OF VELOCITY OF GAS PHASE IN THE PLANE TRAVELING WITH THE MEAN GAS MOTION

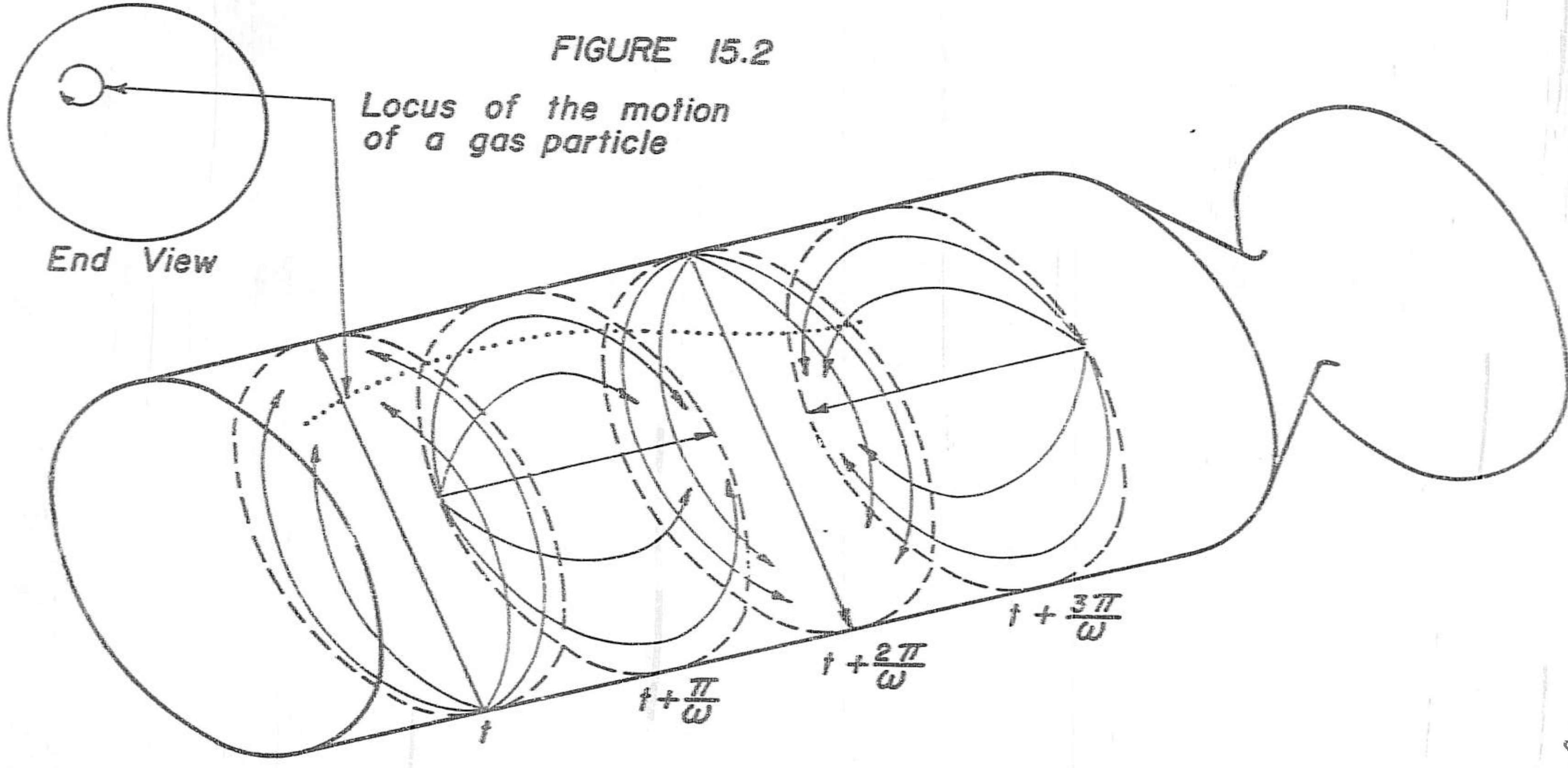
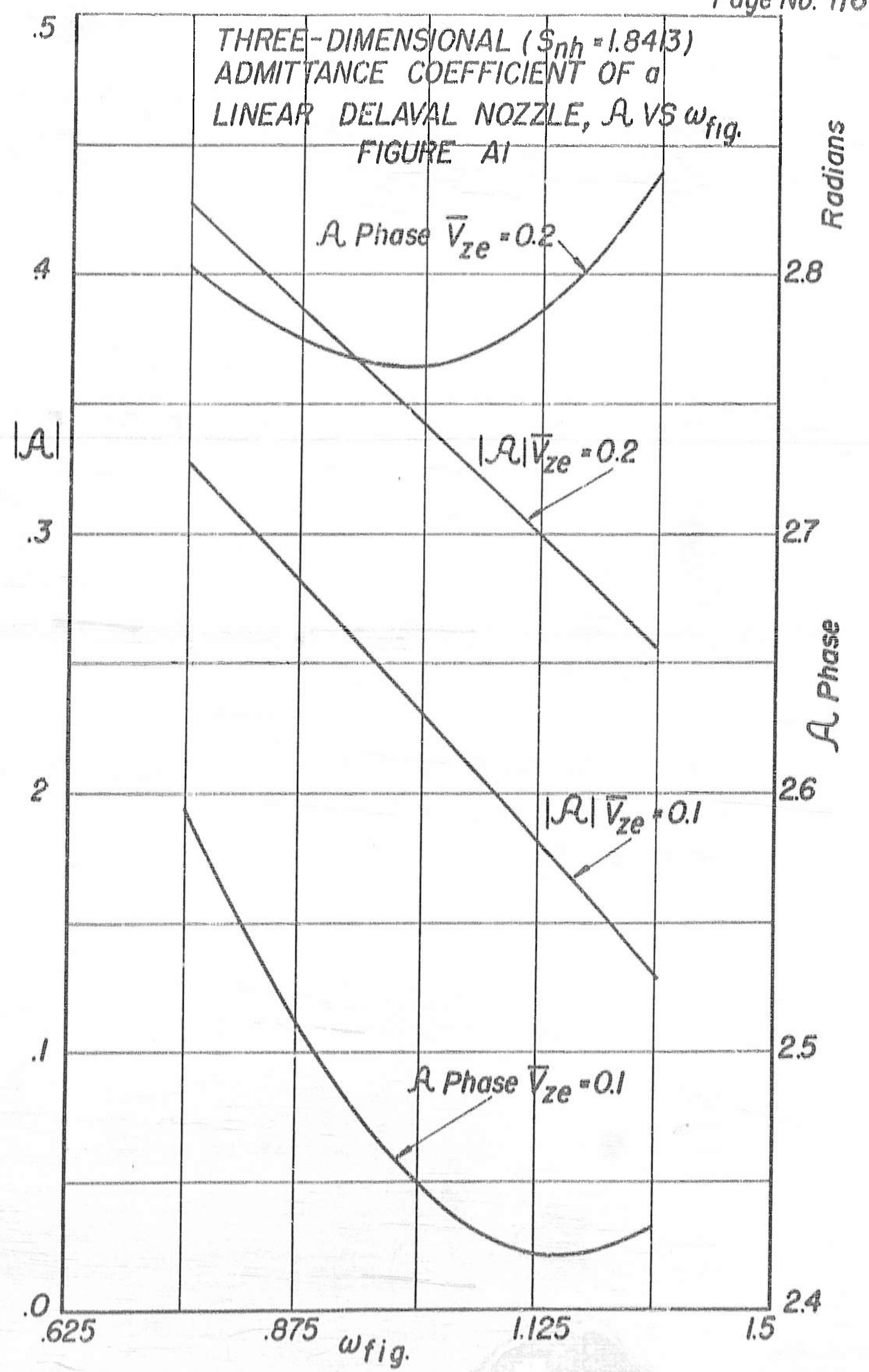
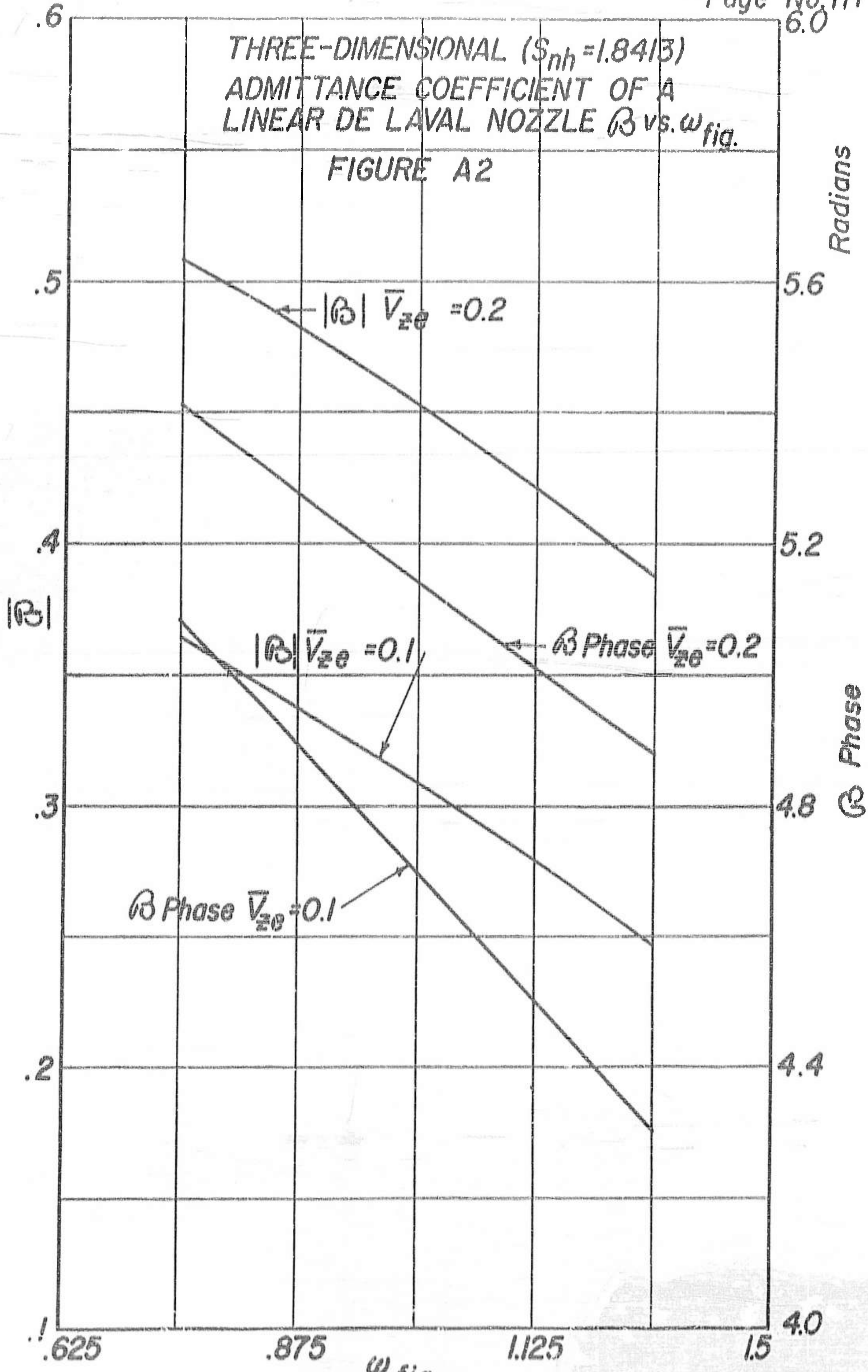
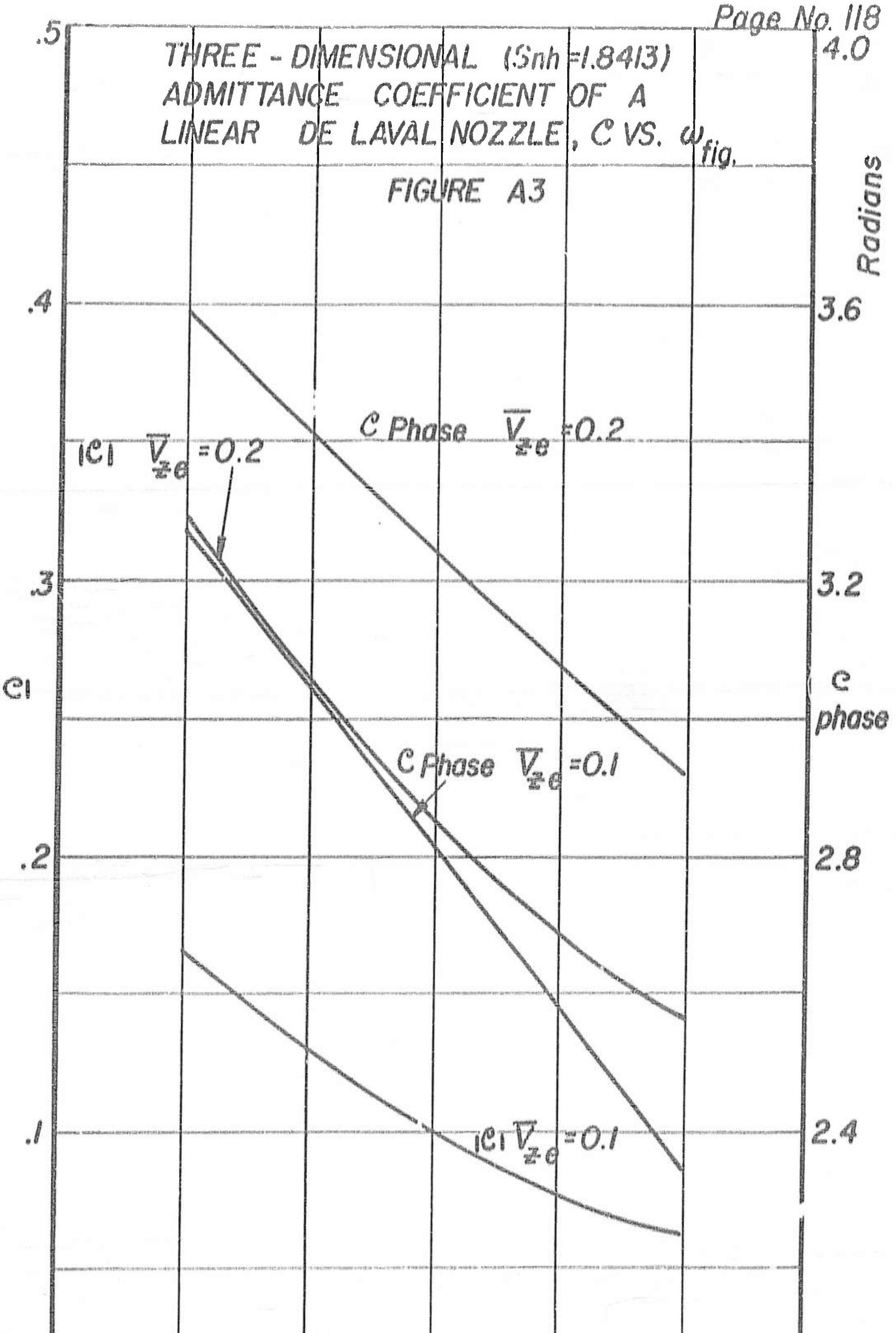
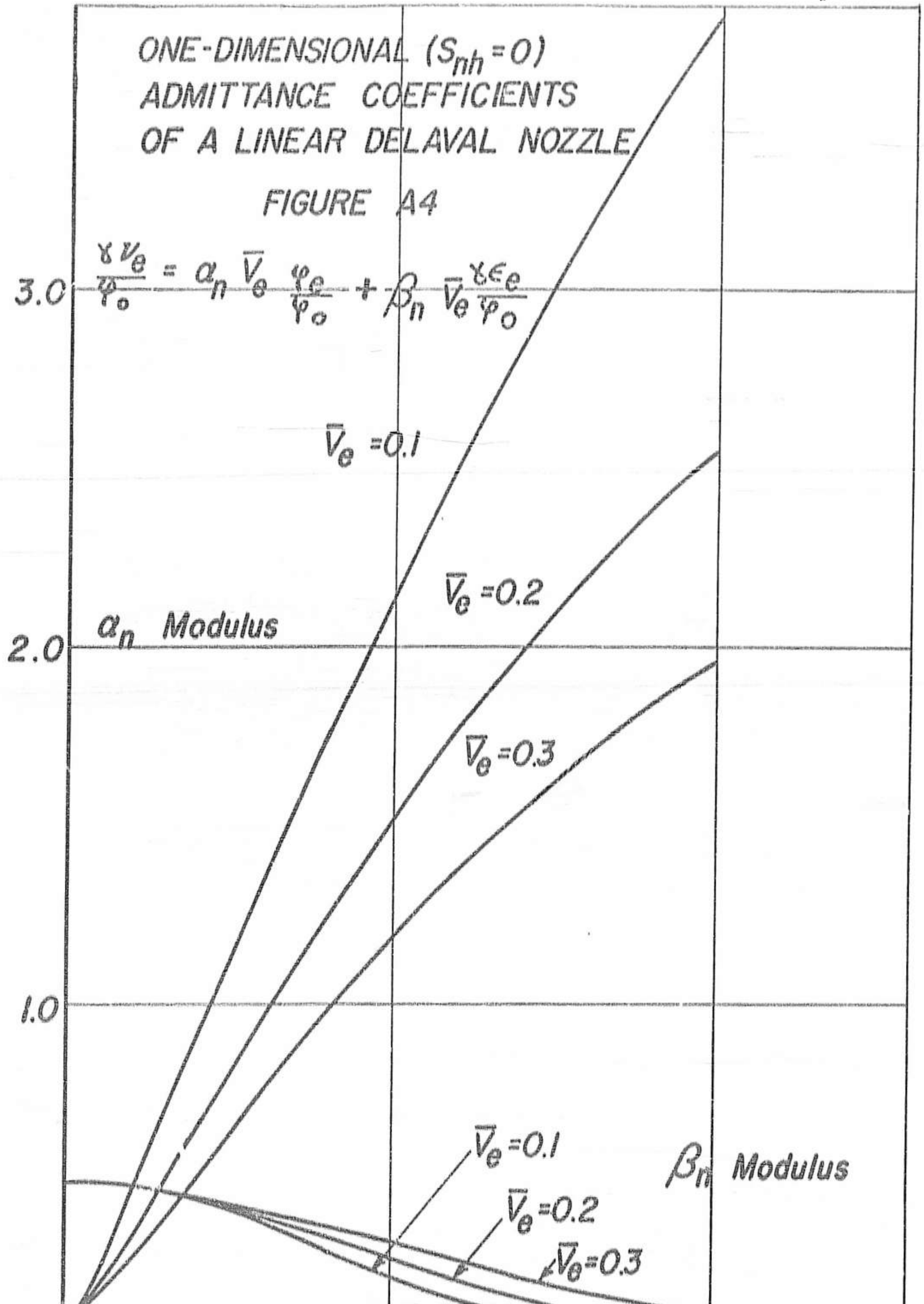


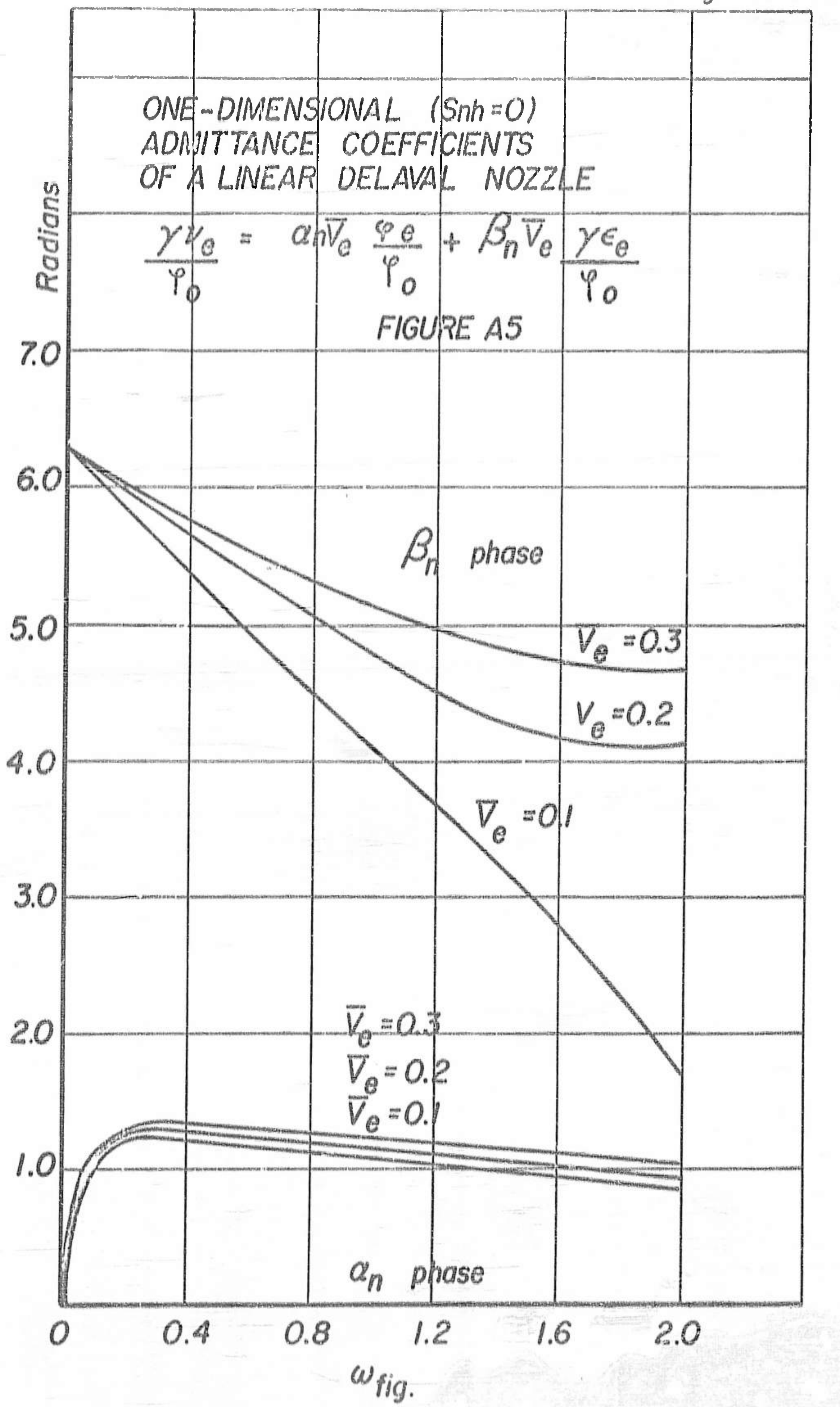
FIGURE 15.2


THREE-DIMENSIONAL (S_{nh} = 1.8413)
ADMITTANCE COEFFICIENT OF a
LINEAR DELAVAL NOZZLE, $\mathcal{A}$ VS $\omega_{fig.}$
FIGURE A1
$\mathcal{A}$ Phase $\bar{V}_{ze}$ = 0.2
$|\mathcal{A}|$ $\bar{V}_{ze}$ = 0.2
$|\mathcal{A}|$ $\bar{V}_{ze}$ = 0.1
$\mathcal{A}$ Phase $\bar{V}_{ze}$ = 0.1
$|\mathcal{A}|$
.5
4
.3
2
.1
.0
Radians
$\mathcal{A}$ Phase
2.8
2.7
2.6
2.5
2.4
.625
.875
1.125
1.5
$\omega_{fig.}$

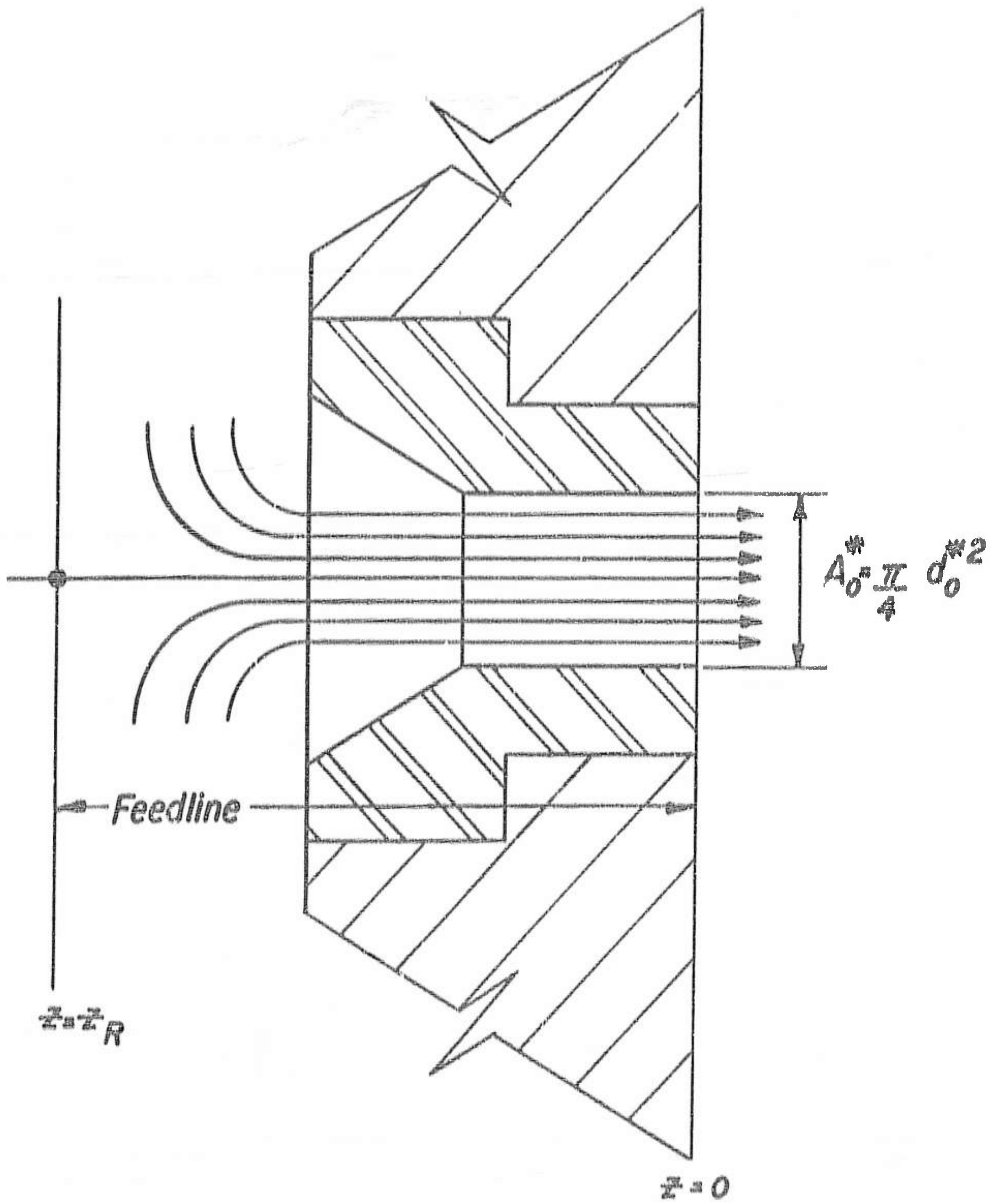

THREE-DIMENSIONAL (S_{nh}=1.8413)
ADMITTANCE COEFFICIENT OF A
LINEAR DE LAVAL NOZZLE ß vs. $\omega_{fig.}$
FIGURE A2
|ß| $\bar{V}_{ze}$ =0.2
|ß| $\bar{V}_{ze}$ =0.1
ß Phase $\bar{V}_{ze}$=0.2
ß Phase $\bar{V}_{ze}$=0.1
|ß|
Radians
ß Phase
ω_{fig}
.6
.5
.4
.3
.2
.1
6.0
5.6
5.2
4.8
4.4
4.0
.625
.875
1.125
1.5


THREE - DIMENSIONAL (Snh = 1.8413)
ADMITTANCE COEFFICIENT OF A
LINEAR DE LAVAL NOZZLE, C VS. $\omega_{fig.}$
FIGURE A3
C Phase $\overline{V}_{ze}$ = 0.2
|C| $\overline{V}_{ze}$ = 0.2
C Phase $\overline{V}_{ze}$ = 0.1
|C| $\overline{V}_{ze}$ = 0.1
|C|
C phase
Radians
.5
.4
.3
.2
.1
0
4.0
3.6
3.2
2.8
2.4
2.0
.625
.875
1.125
1.5
$\omega_{fig.}$

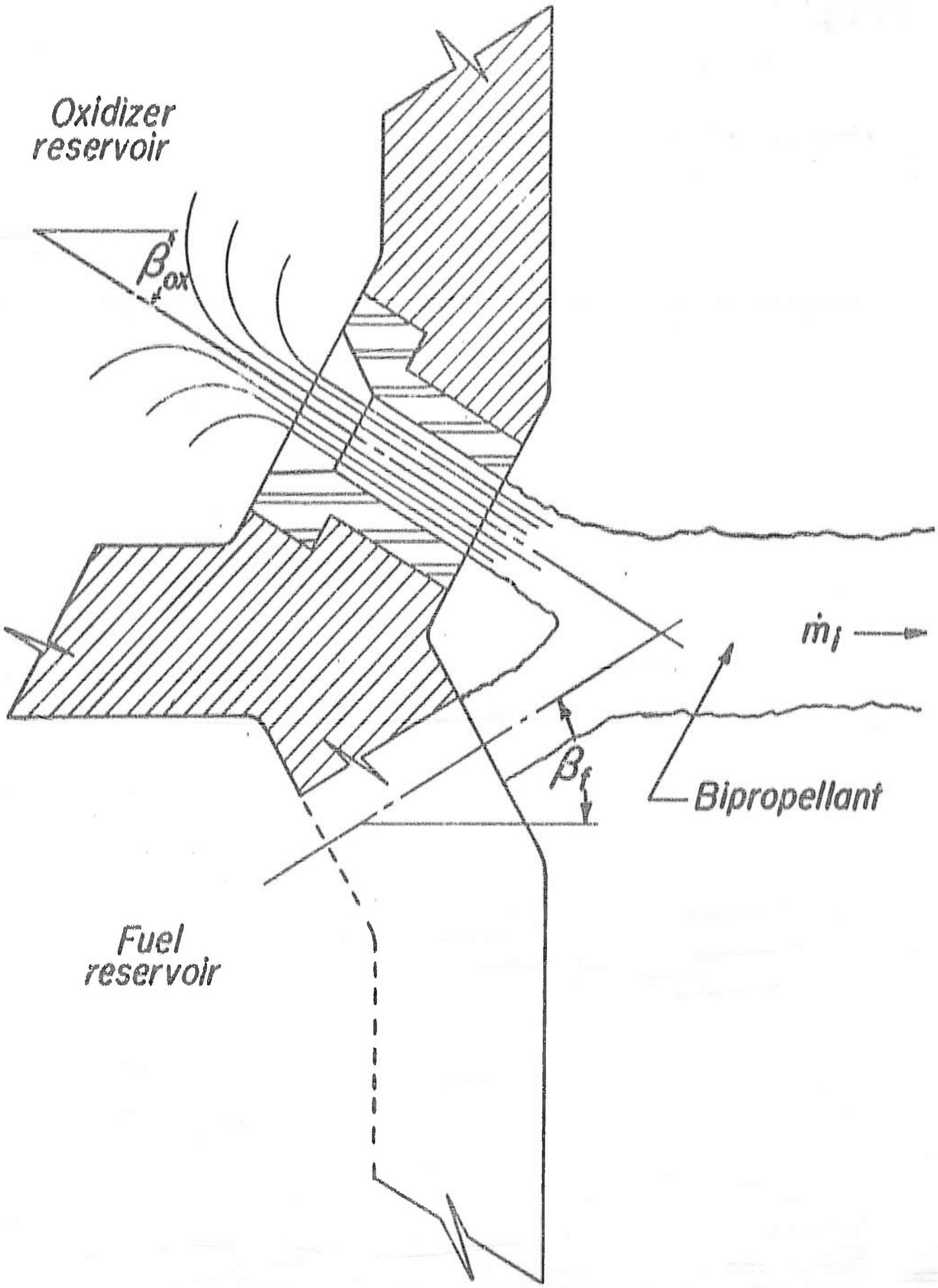

ONE-DIMENSIONAL ($S_{nh}=0$)
ADMITTANCE COEFFICIENTS
OF A LINEAR DELAVAL NOZZLE
FIGURE A4
$\frac{\gamma \nu_e}{\varphi_o} = \alpha_n \bar{V}_e \frac{\varphi_e}{\varphi_o} + \beta_n \bar{V}_e \frac{\gamma \epsilon_e}{\varphi_o}$
3.0
2.0
1.0
0
1.0
2.0
α_n Modulus
$\bar{V}_e = 0.1$
$\bar{V}_e = 0.2$
$\bar{V}_e = 0.3$
β_n Modulus
$\bar{V}_e = 0.1$
$\bar{V}_e = 0.2$
$\bar{V}_e = 0.3$
$\omega_{fig.}$



ONE-DIMENSIONAL (Snh=O)
ADMITTANCE COEFFICIENTS
OF A LINEAR DELAVAL NOZZLE
$\frac{\gamma \nu_e}{\varphi_o} = \alpha_n \bar{V}_e \frac{\varphi_e}{\varphi_o} + \beta_n \bar{V}_e \frac{\gamma \epsilon_e}{\varphi_o}$
FIGURE A5
Radians
7.0
6.0
5.0
4.0
3.0
2.0
1.0
β_n phase
$\bar{V}_e = 0.3$
$\bar{V}_e = 0.2$
$\bar{V}_e = 0.1$
$\bar{V}_e = 0.3$
$\bar{V}_e = 0.2$
$\bar{V}_e = 0.1$
α_n phase
0
0.4
0.8
1.2
1.6
2.0
$\omega_{fig.}$



INJECTION SYSTEM COORDINATE SYSTEM
FIGURE B1



SCHEMATIC OF INJECTION PROCESS
FIGURE B2

APPENDIX A

Supercritical Oscillatory Discharge of Converging-Diverging Nozzles

The oscillatory discharge of converging-diverging nozzle operating in the supercritical range has been treated by Tsien (Ref. 6) and Crocco (Refs. 7 and 19).

Tsien's treatment was one-dimensional and was restricted to the case in which the oscillations in the incoming flow were isothermal. Furthermore, his solutions were obtained for very low frequency and for the asymptotic case of very high frequency. Of these, the latter of the two is beyond the range of applicability to rocket motor oscillations, while the former is not sufficiently general. Therefore, Crocco extended this treatment to the general case of longitudinal and transverse non-isothermal oscillations, and he obtained solutions over the full frequency range of interest.

We have used the results obtained by Crocco as the boundary condition for the combustion chamber analysis, and have written these in two different forms. For the analysis of transverse perturbations, we have taken:

$$\frac{\gamma \nu_{ze}}{\varphi_o} + \mathcal{A} \frac{\varphi_e}{\varphi_o} + \mathcal{B}\, S_{nh} \frac{\gamma \nu_{re}}{\varphi_o} + \mathcal{C} \frac{\gamma \epsilon_e}{\varphi_o} = 0 \qquad 7.7$$

where $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are the nozzle admittance coefficients and are complex functions of the frequency ω , S_{nh}, and the nozzle geometry. This equation states that for a given nozzle, the perturbations at the nozzle entrance must bear a certain relationship to each other with regard to phase and amplitude if neutral oscillations are to be maintained for a given mode and frequency.

For the purely longitudinal case, we have taken:

$$\frac{\gamma \nu_e}{\varphi_o} = \alpha_n \bar{V}_e \frac{\varphi_e}{\varphi_o} + \beta_n \bar{V}_e \frac{\gamma \epsilon_e}{\varphi_o} \qquad 12.14$$

It is clear that when $S_{nh} = 0$, we are treating the purely longitudinal case

and Eq. 7.7 must reduce to Eq. 12.14. For one-dimensional flow, the subscript z is no longer necessary and then we obtain:

$$\mathcal{A} = -\alpha_n \bar{V}_e \quad ; \quad \mathcal{C} = -\beta_n \bar{V}_e \qquad \text{A 1}$$

It is of some interest to consider the admittances α_n and β_n in greater detail. In Reference 6, Tsien has shown that for nozzles which have a linear variation of velocity in their subsonic part, the governing equations is a non-homogeneous hypergeometric differential equation in the form:

$$y(1-y)\frac{d^2\sigma}{dy^2} - \left(2 + \frac{2i\beta}{\gamma+1}\right) y \frac{d\sigma}{dy} - \frac{i\beta}{2}\frac{(2+i\beta)}{(\gamma+1)}\sigma =$$

$$-i\beta\epsilon_e\left(\frac{y}{y_e}\right)^{-\frac{i\beta}{2}}\left[\frac{1-i\beta\frac{\gamma-1}{2\gamma}}{2(\gamma+1)} + \frac{2+i\beta}{4\gamma}\frac{1}{y}\right] \qquad \text{A 2}$$

where $y = \dfrac{\bar{V}^*}{\bar{c}_{th}^*}$ so that at the nozzle entrance $y_e = \frac{\gamma+1}{2}\bar{V}_e^2$ and β is a non-dimensional frequency $\beta = \omega f_{ig}$ which may be related to the ω of the chamber which appears in all the chamber equations, and hence ωf_{ig} will be used as the independent variable for all the curves of the admittances.

The relevant condition to be satisfied at the nozzle entrance is given by Crocco (Ref. 7) as:

$$\varphi_e - \gamma\sigma_e = \gamma\epsilon_e \qquad \text{A 3}$$

In this same reference, a series solution was obtained for low frequency by taking:

$$\sigma(y,\beta) = \sigma^{(0)}(y) + i\beta\sigma^{(1)}(y) + (i\beta)^2\sigma^{(2)}(y) + \cdots$$
$$\nu(y,\beta) = \nu^{(0)}(y) + i\beta\nu^{(1)}(y) + (i\beta)^2\nu^{(2)}(y) + \cdots \qquad \text{A 4}$$

however, the series was cut off after only two terms and the results were correct to $O(i\beta)$. Since that reference first appeared, some exact numerical solutions have been obtained over a wide frequency range, and direct comparison shows that α_n and β_n may be determined analytically with sufficient

accuracy when $\omega_{fig} < 0.2$ by first calculating an additional term in the series. When this is done, there results:

$$\frac{\gamma \nu_e}{\varphi_o} = \frac{\gamma-1}{2}\left[1 - i\beta\left(\frac{1}{2} + \frac{1}{\gamma-1}\frac{\log y_e}{1-y_e}\right) + (i\beta)^2\left\{\frac{1}{4} + \frac{1}{2(\gamma+1)}\frac{y_e}{1-y_e}\right.\right.$$

$$\left.\left. - \frac{2}{(\gamma^2-1)(1-y_e)}\sum_{n=1}^{\infty}\frac{1-y_e^n}{n^2} - \frac{1}{2(\gamma+1)(1-y_e)} + \frac{\log y_e}{(\gamma^2-1)(1-y_e)}\left[2\log y_e + \frac{y_e \log y_e}{1-y_e}\right]\right\}\right]\bar{V}_e \frac{\varphi_e}{\varphi_o}$$

$$+ \frac{1}{2}\left[1 + \frac{i\beta}{2}\frac{\log y_e}{1-y_e} + (i\beta)^2\left\{\frac{1}{(\gamma+1)(1-y_e)}\sum_{n=1}^{\infty}\frac{1-y_e^n}{n^2} - \frac{(\gamma-1)\,y_e}{4(\gamma+1)(1-y_e)}\right.\right. \qquad \text{A 5}$$

$$\left.\left. + \frac{\gamma-1}{4(\gamma+1)(1-y_e)} - \frac{\log y_e}{(\gamma+1)(1-y_e)}\left[\log(1-y_e) + \frac{y_e \log y_e}{2(1-y_e)} + \frac{\gamma+1}{8}\log y_e\right]\right\}\right]\bar{V}_e \frac{\gamma \epsilon_e}{\varphi_o}$$

$$= \alpha_n \bar{V}_e \frac{\varphi}{\varphi_o} + \beta_n \bar{V}_e \frac{\gamma \epsilon_e}{\varphi_o}$$

For example, taking $\gamma = 1.20$ and $\bar{V}_e = 0.10$, we obtain:

$$\alpha_n = .1\left[1 + 22.300 i\beta + 6.883\beta^2\right] \qquad \text{A 6}$$

$$\beta_n = .5\left[1 - 2.280 i\beta - 1.883\beta^2\right]$$

As stated previously, all graphs of α_n and β_n are plotted utilizing ω_{fig} as the independent variable. This quantity is related to the frequency in the chamber by means of:

$$\frac{\omega_{fig}}{\omega} = \frac{\beta}{\omega} = \frac{r^*_{th}}{K L^*}\sqrt{\frac{\gamma+1}{2}} \qquad \text{A 7}$$

where K is the non-dimensional velocity gradient in the subsonic portion of the nozzle:

$$K = \frac{r^*_{th}}{\bar{c}^*_{th}}\frac{d\bar{V}^*}{dz^*} \qquad \text{A 8}$$

Thus, the smaller K is, the longer is the subsonic portion of the nozzle, since the length of the subsonic portion of the nozzle is given by:

$$l^*_{sub} = \frac{r^*_{th}}{K}\left[1 - \sqrt{\frac{\gamma+1}{2}}\,\bar{V}_e\right] \qquad \text{A 9}$$

The graphs of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are likewise plotted versus ω_{fig}, but in this case, the frequency in the chamber ω is related to ω_{fig} by:

$$\frac{\omega_{fig}}{\omega} = \frac{r^*_{th}}{K r_c^*} \sqrt{\frac{\gamma+1}{2}} \qquad \text{A 10}$$

since another scheme of non-dimensionalization was utilized in these computations.

APPENDIX B

Derivation of Injector Response

We will determine here the variation in mixing ratio and injection velocity as a function of the oscillating pressure at the injector face. This will be done by first considering the frequency response of a single feedline and then selecting the desired impingement pattern to yield the combined effect of propellant and oxidizer. The relevant equations for quasi-one-dimensional flow in the feedline are:

$$\frac{\partial}{\partial t^*}\left(\rho_\ell^* A^*\right) + \frac{\partial}{\partial z^*}\left(\rho_\ell^* A^* V_\ell^*\right) = 0 \qquad \text{B 1}$$

$$\frac{\partial V_\ell^*}{\partial t^*} + V_\ell^* \frac{\partial V_\ell^*}{\partial z^*} = -\frac{1}{\rho_\ell^*}\frac{\partial p^*}{\partial z^*} \qquad \text{B 2}$$

which are the conservation of mass and momentum respectively, and where A^* is the cross-section of the feedline. The coordinate system is indicated in Fig. B1.

Upon non-dimensionalizing as follows:

$$V_\ell = \frac{V_\ell^*}{\bar{c}_o^*}, \quad t = \frac{\bar{c}_o^* t^*}{L^*}, \quad p = \frac{p^*}{\bar{p}_o^*}, \quad \rho_\ell = \frac{\rho_\ell^*}{\bar{\rho}_o^*}$$

$$A = \frac{A^*}{L^{*2}}, \quad z = \frac{z^*}{L^*}, \quad \bar{c}_o^* = \sqrt{\frac{\gamma\, \bar{p}_o^*}{\bar{\rho}_o^*}} \qquad \text{B 3}$$

where L^* is the chamber length, we obtain:

$$\frac{\partial}{\partial z}\left(\rho_\ell A V_\ell\right) = \frac{\partial}{\partial z}\left(\dot{m}\right) = 0 \qquad \text{B 1a}$$

$$\frac{\partial V_\ell}{\partial t} + V_\ell \frac{\partial V_\ell}{\partial z} = -\frac{1}{\gamma \rho_\ell}\frac{\partial p}{\partial z} \qquad \text{B 2a}$$

Introducing small perturbations,

$$V_\ell(z,t) = \bar{V}_\ell(z) + V_\ell'(z,t) \qquad \text{B 3}$$

$$p(z,t) = \bar{p}(z) + p'(z,t)$$

continuity and momentum yield:

$$\frac{d}{dz}\left(\rho_\ell A \bar{V}_\ell\right) = \frac{d}{dz}\left(\bar{\dot{m}}\right) = 0 \qquad \text{B 1b}$$

$$\frac{\partial}{\partial z}\left(\rho_\ell A V_\ell'\right) = \frac{\partial}{\partial z}\left(\dot{m}'\right) = 0 \qquad \text{B 1c}$$

$$\bar{V}_\ell \frac{d\bar{V}_\ell}{dz} = -\frac{1}{\gamma\rho_\ell}\frac{d\bar{p}}{dz} \qquad \text{B 2b}$$

$$\frac{\partial V_\ell'}{\partial t} + \frac{\partial}{\partial z}\left(\bar{V}_\ell V_\ell'\right) = -\frac{1}{\gamma\rho_\ell}\frac{\partial p'}{\partial z} \qquad \text{B 2c}$$

We may separate the variables by taking:

$$p'(z,t) = \varphi(z)e^{st}$$
$$V_\ell'(z,t) = \eta(z)e^{st} \qquad \text{B 4}$$

and then we obtain:

$$\frac{d}{dz}\left(\rho_\ell A\eta\right) = 0 \qquad \text{B 1d}$$

$$s\eta + \frac{d}{dz}\left(\bar{V}_\ell\eta\right) = -\frac{1}{\gamma\rho_\ell}\frac{d\varphi}{dz} \qquad \text{B 2d}$$

Before solving these equations, let us briefly consider the boundary conditions. At the injector face, ($z = 0$), the steady state pressure and injection velocity have their design values while the pressure perturbation must be the chamber pressure perturbation, and the as yet unknown velocity perturbation will correspond to an initial condition for the chamber. Because of the relatively small diameter of conventional injection ports with regard to the reservoir proper, which means that only a small liquid inertia is involved, and because of the further likelihood of entrapped vapor in the reservoir, it seems reasonable to assume that the reservoir pressure P_R is constant only a short distance from the port. That is, the reservoir proper, is an infinite source of liquid and the velocity vanishes where the pressure is constant. Hence, we may summarize the boundary conditions as:

$$z = 0: \quad \bar{V}_\ell(0) = \bar{V}_{\ell 0} \qquad \bar{p}(0) = 1$$
$$\eta(0) = \eta_0 \qquad \varphi(0) = \varphi_0 \qquad \text{B 5}$$
$$z = z_R: \quad \bar{V}_\ell(z_R) = 0 \qquad \bar{p}(z_R) = \bar{P}_R$$
$$\eta(z_R) = 0 \qquad \varphi(z_R) = 0$$

Integrating Eq. B 2b and applying the boundary conditions, we obtain:

$$\bar{P}_R = 1 + \frac{1}{2}\gamma \rho_\ell \bar{V}_{\ell o}^{\,2} \qquad \text{B 6}$$

Eqs. B 1b and B 1c yield:

$$\rho_\ell A \bar{V}_\ell = \bar{\dot{m}} = \text{const.}$$

$$\rho_\ell A \eta = \dot{m}' = \text{const.}$$

or

$$\frac{\eta(z)}{\bar{V}_\ell(z)} = \text{const.} = \frac{\eta_o}{\bar{V}_{\ell o}} \qquad \text{B 7}$$

Now integrating Eq. B 2d from z_R to $z = 0$, we obtain:

$$s\int_{z_R}^{0} \eta(z)\,dz + \int_{z_R}^{0} d(\bar{V}_\ell \eta) = -\frac{1}{\gamma\rho_\ell}\int_{z_R}^{0} d\varphi \qquad \text{B 8}$$

On introducing Eq. B 7, the first term yields:

$$\int_{z_R}^{0} \eta(z)\,dz = \eta_o A_o \int_{z_R}^{0} \frac{dz}{A(z)} = \frac{\eta_o A_o}{L_{eq.}} \qquad \text{B 9}$$

where $L_{eq} = \left[\int_{z_R}^{0} \frac{dz}{A(z)}\right]^{-1}$ is an equivalent length.

Substituting back and solving for η_o, we find:

$$\eta_o = \frac{-1}{\gamma\rho_\ell \bar{V}_{\ell o}(1+sa)}\,\varphi_o \qquad \text{B 10}$$

where the quantity

$$a = \frac{A_o}{\bar{V}_{\ell o} L_{eq.}} \qquad \text{B 11}$$

is a dimensionless characteristic time of the feedline.

We may define the transfer function:

$$g = \frac{\eta_o}{\varphi_o} = \frac{-1}{\gamma\rho_\ell \bar{V}_{\ell o}(1+sa)} \qquad \text{B 12}$$

which, for neutral oscillations, becomes:

$$g(w) = \frac{-e^{-i\tan^{-1} wa}}{\gamma\rho_\ell \bar{V}_{\ell o}\sqrt{1+w^2a^2}} \qquad \text{B 13}$$

And now, let us use this result in deriving the response of a typical bipropellant injector where we will let the subscripts ox. and f denote oxidizer and fuel respectively. If we consider an injector of the

type shown in Fig. B2, then the longitudinal component of injection velocity is:

$$\bar{V}_{\ell o} = (\bar{V}_{\ell o}\cos\beta)_{ox} + (\bar{V}_{\ell o}\cos\beta)_f \qquad \text{B 14}$$

where we have assumed conservation of the axial component of momentum during impingement in the steady state. And for unsteady flow, under the same assumption,

$$V_{\ell o}' = \left\{ (g(\omega)\cos\beta)_{ox} + (g(\omega)\cos\beta)_f \right\} \varphi_o e^{st} \qquad \text{B 15}$$

where we have assumed that the fuel and oxidizer ports experience the same instantaneous chamber pressure.

Proceeding, we let r denote the mixture ratio, then

$$r = \bar{r} + r' = \frac{\dot{m}_{ox}}{\dot{m}_f} = \frac{\bar{\dot{m}}_{ox}(1+\mu_{ox})}{\bar{\dot{m}}_f(1+\mu_f)} \cong \bar{r}(1+\mu_{ox}-\mu_f) \qquad \text{B 16}$$

where we have neglected second order terms and where $\mu = \frac{\dot{m}'}{\bar{\dot{m}}}$ denotes the fractional perturbation in mass flow. Now,

$$\mu_o(t) = \frac{\dot{m}_o'}{\bar{\dot{m}}_o} = \frac{\rho_\ell V_{\ell o}' A_o}{\rho_\ell \bar{V}_{\ell o} A_o} = \frac{V_{\ell o}'}{\bar{V}_{\ell o}} = \frac{g(\omega)}{\bar{V}_{\ell o}}\varphi_o e^{st} \qquad \text{B 17}$$

and hence substituting into Eq. B 16 we obtain:

$$\left(\frac{r'}{\bar{r}}\right)_o = (\mu_{ox}-\mu_f)_o = \left[\left(\frac{g(\omega)}{\bar{V}_{\ell o}}\right)_{ox} - \left(\frac{g(\omega)}{\bar{V}_{\ell o}}\right)_f\right]\varphi_o e^{st} \qquad \text{B 18}$$

We may now obtain a relation for the fractional perturbation in the injection rate. Since,

$$\dot{m}_i = \bar{\dot{m}}_i + \dot{m}_i' = \bar{\dot{m}}_{ox}(1+\mu_{ox}) + \bar{\dot{m}}_f(1+\mu_f)$$

there follows

$$\frac{\dot{m}_i'}{\bar{\dot{m}}_i} = \frac{\bar{r}}{1+\bar{r}}\mu_{ox} + \frac{1}{1+\bar{r}}\mu_f \qquad \text{B 19}$$

and on introducing Eq. B 17

$$\left(\frac{\dot{m}_i'}{\bar{\dot{m}}_i}\right)_o = \left\{\frac{\bar{r}}{1+\bar{r}}\left(\frac{g(\omega)}{\bar{V}_{\ell o}}\right)_{ox} + \frac{1}{1+\bar{r}}\left(\frac{g(\omega)}{\bar{V}_{\ell o}}\right)_f\right\}\varphi_o e^{st} \qquad \text{B 20}$$

and we note that although $\frac{\dot{m}_o'}{\dot{m}_o} = \frac{V_{\ell o}'}{V_{\ell o}}$ for each feedline, we cannot put Eq. B 15 into the same form as Eq. B 20, since in general $(\cos\beta)_{ox}$ and $(\cos\beta)_f$ are not equal. Thus

$$\left(\frac{V_{\ell o}'}{V_{\ell o}}\right) = \left\{ \frac{(g(w)\cos\beta)_{ox} + (g(w)\cos\beta)_f}{(\bar{V}_{\ell o}\cos\beta)_{ox} + (\bar{V}_{\ell o}\cos\beta)_f} \right\} \varphi_o e^{st} \qquad \text{B 21}$$

Hence the transfer function for the case of a bipropellant injector is given by

$$\frac{\eta_o}{\varphi_o} = J(w)\,\bar{V}_{\ell o} \qquad \text{B 22}$$

where

$$J(w) = \left\{ \frac{(g(w)\cos\beta)_{ox} + (g(w)\cos\beta)_f}{(\bar{V}_{\ell o}\cos\beta)_{ox} + (\bar{V}_{\ell o}\cos\beta)_f} \right\} \qquad \text{B 23}$$

We may summarize these results by writing:

$$\begin{aligned} \left(\frac{F'}{F}\right)_o &= G(w)\,\varphi_o e^{st} \\ \left(\frac{\dot{m}_i'}{\dot{m}_i}\right)_o &= H(w)\,\varphi_o e^{st} \\ \left(\frac{V_{\ell o}'}{\bar{V}_{\ell o}}\right)_o &= J(w)\,\varphi_o e^{st} \end{aligned} \qquad \text{B 24}$$

where $G(w)$, and $H(w)$ are defined through Eqs. B 18 and B 20. We note that if another injection system is utilized, then the analytical form of the injector response is nevertheless given by equations of the form B 24.

One additional quantity will be derived here. We have stated elsewhere that we will assume that $\frac{dh_{\ell s}}{dt} = 0$, and we may therefore observe that each particle will retain its initial value of $h_{\ell s}'$, $\bar{h}_{\ell s}$ being a constant. This enables us to write:

$$\begin{aligned} h_{\ell s}'(z,t) &= h_{\ell s}'\left(0,\, t - \int_0^z \frac{dz'}{V_\ell[z', t'(z')]}\right) \\ &\cong h_{\ell s o}'\left(t - \int_0^z \frac{dz'}{\bar{V}_\ell(z')}\right) \\ &= h_{\ell o}'\left(t - \int_0^z \frac{dz'}{\bar{V}_\ell(z')}\right) + (\gamma-1)\bar{V}_{\ell o}V_{\ell o}'\left(t - \int_0^z \frac{dz'}{\bar{V}_\ell(z')}\right) \end{aligned} \qquad \text{B 25}$$

The specific enthalpy h_{ℓ_0} of the propellants depends on the instantaneous value of the mixture ratio and hence the enthalpy will be larger or smaller depending on the mixture ratio which in turn depends on the pressure at the chamber face. Expanding h_ℓ in a Taylor series about its steady state value, we obtain:

$$h_\ell = \bar{h}_\ell + \frac{dh_\ell}{dr}(r - \bar{r}) + \cdots\cdots$$

and defining

$$2K = \frac{\bar{r}}{\bar{h}_{\ell_0}}\left(\overline{\frac{dh_\ell}{dr}}\right)_0 \qquad \text{B 26}$$

we obtain

$$\left(\frac{h_\ell'}{\bar{h}_\ell}\right)_0 = 2K\left(\frac{r'}{\bar{r}}\right)_0 \qquad \text{B 27}$$

Thus Eq. B 25 becomes:

$$h_{\ell s}'(z,t) = 2K\bar{h}_\ell\left(\frac{r'}{\bar{r}}\right)_0\left[t - \int_0^z \frac{dz'}{\bar{V}_\ell(z')}\right] + (\gamma-1)\bar{V}_{\ell_0}^2\left(\frac{V_{\ell_0}'}{\bar{V}_{\ell_0}}\right)\left[t - \int_0^z \frac{dz'}{\bar{V}_\ell(z')}\right] \qquad \text{B 28}$$

and substituting from Eq. B 24 we obtain:

$$\begin{aligned} h_{\ell s}'(z,t) &= \left[2K\bar{h}_\ell G(\omega) + (\gamma-1)\bar{V}_{\ell_0}^2 J(\omega)\right]\varphi_0 e^{s\left\{t - \int_0^z \frac{dz'}{\bar{V}_\ell(z')}\right\}} \\ &= M(\omega)\,\varphi_0\, e^{s\left\{t - \int_0^z \frac{dz'}{\bar{V}_\ell(z')}\right\}} \end{aligned} \qquad \text{B 29}$$

which defines M (W).

Appendix C

Derivation of Equation 13.1

We will begin by introducing Eq. 12.15 into Eq. 12.16, and then setting $\bar{\tau}(z') = \bar{\delta}$ (constant), there follows:

$$\begin{aligned}\left(\frac{\gamma \nu_e}{\varphi_0}\right)^{(1)} &= \gamma n\left(1 - e^{-i\omega\bar{\delta}}\right)\int_0^1 \cos\omega z' \frac{d\bar{V}}{dz'}(z')dz' \\ &+ \gamma\left\{H(\omega) - m\bar{\delta}\, i\omega\, G(\omega)\right\}\int_0^1 e^{-i\omega\bar{\tau}_t(z')}\frac{d\bar{V}}{dz'}(z')dz' \\ &- \gamma\bar{V}_e\cos\omega - i\sin\omega + i\omega\int_0^1\left[\gamma E^{(0)}(z')\cos\omega(1-z')\right]dz' \\ &- \omega\int_0^1\left[\gamma F^{(0)}(z')\sin\omega(1-z')\right]dz'\end{aligned} \qquad \text{C 1}$$

which may be rewritten as

$$\begin{aligned}\left(\frac{\gamma \nu_e}{\varphi_0}\right)^{(1)} &= -i\sin\omega - \gamma\bar{V}_e\cos\omega + \Pi\cdot L(\omega) \\ &+ \boxed{H}\cdot D(\omega) + B(\omega)\end{aligned} \qquad \text{C 2}$$

where we have introduced the notation

$$\begin{aligned}\Pi &= \gamma\left\{H(\omega) - m\bar{\delta}\, i\omega\, G(\omega)\right\} \\ L(\omega) &= \int_0^1 e^{-i\omega\bar{\tau}_t(z')}\frac{d\bar{V}}{dz'}(z')dz' \\ \boxed{H} &= \gamma n\left(1 - e^{-i\omega\bar{\delta}}\right)\end{aligned} \qquad \text{C 3}$$

and upon integration by parts, it may be shown that

$$\begin{aligned}D(\omega) &= \int_0^1 \cos\omega(1-z')\cos\omega z' \frac{d\bar{V}}{dz'}(z')dz' \\ &= \bar{V}_e\cos\omega - \omega\int_0^1\frac{d\bar{V}}{dz'}(z')\sin\omega(1-2z')dz'\end{aligned} \qquad \text{C 4}$$

while $B(\omega)$ consists of the terms:

$$B(\omega) = \omega \Big\{ (\gamma-1)\int_0^1 \bar{V}(z')\cos\omega z' \sin\omega(1-z')\,dz'$$

$$- \Pi \int_0^1 \sin\omega(1-z')dz' \int_0^{z'} e^{-i\omega\bar{\tau}_t(z'')} \frac{d\bar{V}}{dz''}(z'')\,dz''$$

$$- \int_0^1 \frac{\gamma q^{(0)}}{\varphi_0}(z')\left[1 - \frac{ik}{\omega}\bar{\rho}_\ell(z')\right] \sin\omega(1-z')\,dz'$$

$$+ \frac{k}{\omega}\int_0^1 \bar{\rho}_\ell(z')\sin\omega z' \sin\omega(1-z')\,dz'$$

$$- \frac{i\gamma}{\omega}\int_0^1 \bar{V}_\ell(z')\frac{1}{\varphi_0}\frac{dq^{(0)}}{dz'}(z')\sin\omega(1-z')\,dz' \qquad \text{C 5}$$

$$+ (3-\gamma)\int_0^1 \bar{V}(z')\frac{i\gamma q^{(0)}(z')}{\varphi_0}\cos\omega(1-z')\,dz'$$

$$+ (3-\gamma)\int_0^1 \bar{V}(z')\sin\omega z' \cos\omega(1-z')\,dz'$$

$$+ \gamma M(\omega)\int_0^1 e^{-i\omega\bar{\tau}_t(z')}\frac{d\bar{V}}{dz'}(z')\cos\omega(1-z')\,dz' \Big\}$$

The first, fourth and seventh terms may be expanded and then combined to yield:

$$\beta_B = \omega\sin\omega\int_0^1 \bar{V}(z')dz' - (2-\gamma)\omega\int_0^1 \bar{V}(z')\sin\omega(1-2z')dz'$$

$$- \frac{k}{2}\int_0^1 \bar{\rho}_\ell(z')\left[\cos\omega - \cos\omega(1-2z')\right]dz' \qquad \text{C 6}$$

If we now define:

$$A(z) = -\gamma\bar{V}_\ell(z)\frac{d\bar{V}}{dz}(z)\,\bar{V}_{\ell_0}^{\,2}\int_0^z \frac{e^{-k\bar{\tau}_t(z')}}{\bar{V}_\ell^{\,3}(z')} \qquad \text{C 7}$$

then from Eq. 12.12 we obtain:

$$\frac{\gamma q^{(0)}}{\varphi_0}(z) = J(\omega)e^{-i\omega\bar{\tau}_t(z)}A(z) \qquad \text{C 8}$$

and

$$\frac{1}{\varphi_0}\frac{d}{dz}\left(\frac{\gamma q^{(0)}}{\varphi_0}(z)\right) = J(\omega)e^{-i\omega\bar{\tau}_t(z)}\left[\frac{dA}{dz}(z) - \frac{A(z)\,i\omega}{\bar{V}_\ell(z)}\right] \qquad \text{C 9}$$

since $$\frac{d}{dt}\bar{\tau}_t(z) = \frac{d}{dz}\int_0^z \frac{dz'}{\bar{V}_\ell(z')} = \frac{1}{\bar{V}_\ell(z)}$$

so that upon introducing the notation

$$B_1(w) = \int_0^1 \bar{\rho}_\ell(z')\cos w(1-2z')\,dz'$$

$$B_2(w) = \int_0^1 \sin w(1-z')\,dz' \int_0^{z'} e^{-iw\bar{\tau}_t(z'')}\frac{d\bar{V}}{dz''}(z'')\,dz''$$

$$B_3(w) = \int_0^1 \bar{V}(z')\sin w(1-2z')\,dz'$$

$$B_4(w) = \int_0^1 e^{-iw\bar{\tau}_t(z')}\frac{d\bar{V}}{dz'}(z')\cos w(1-z')\,dz' \qquad \text{C 10}$$

$$B_5(w) = \int_0^1 A(z')\,e^{-iw\bar{\tau}_t(z')}\sin w(1-z')\,dz'$$

$$B_6(w) = \int_0^1 A(z')\,e^{-iw\bar{\tau}_t(z')}\bar{\rho}_\ell(z')\sin w(1-z')\,dz'$$

$$B_7(w) = \int_0^1 A(z')\,e^{-iw\bar{\tau}_t(z')}\bar{V}(z')\cos w(1-z')\,dz'$$

$$B_8(w) = \int_0^1 \frac{dA}{dz'}(z')\,e^{-iw\bar{\tau}_t(z')}\bar{V}_\ell(z')\sin w(1-z')\,dz'$$

Eq. C 6 becomes:

$$\mathfrak{B}_B = w\sin w\int_0^1 \bar{V}(z')\,dz' - (2-\gamma)w\,B_3(w)$$
$$- \cos w\,\frac{k}{2}\int_0^1 \bar{\rho}_\ell(z')\,dz' + \frac{k}{2}B_1(w) \qquad \text{C 11}$$

and then Eq. C 5 may be written:

$$B(w) = \mathfrak{B}_B - \gamma w B_2(w)H(w) + \gamma w^2 B_2(w)G(w)\,i m\bar{\delta}$$
$$+ J(w)\left[-2wB_5(w) + ikB_6(w) + (3-\gamma)iwB_7(w)\right. \qquad \text{C 12}$$
$$\left. - iB_8(w)\right] + \gamma w M(w) B_4(w)$$

We may treat Eq. 12.17 in a similar manner. First substitute Eq. 12.15 into Eq. 12.17 and set $\bar{\tau}(z') = \bar{\delta}$ (constant). Then we obtain the result:

$$\frac{\varphi_e}{\varphi_0} = \cos w + 2i\bar{V}_e \sin w + w\int_0^1 \left[\gamma E^{(0)}(z') \sin w(1-z')\right] dz' - iw\int_0^1 \left[\gamma F^{(0)}(z') \cos w(1-z')\right] dz' \qquad \text{C 13}$$

which may be rewritten as:

$$\left(\frac{\varphi_e}{\varphi_0}\right) = \cos w + 2i\bar{V}_e \sin w - i\,\boxed{H}\,C(w) + i A(w) \qquad \text{C 14}$$

where upon integration by parts, it may be shown that:

$$C(w) = w\int_0^1 \bar{V}(z') \cos w(1-z')\, dz' \qquad \text{C 15}$$

and $A(w)$ consists of the terms:

$$\begin{aligned} A(w) = w \Big\{ & (\gamma-1)\int_0^1 \bar{V}(z') \cos wz' \cos w(1-z')\, dz' \\ & - \pi\int_0^1 \cos w(1-z')\, dz' \int_0^{z'} e^{-iw\bar{\tau}_t(z'')} \frac{d\bar{V}}{dz''}(z'')\, dz'' \\ & - \int_0^1 \frac{\gamma g^{(0)}}{\varphi_0}(z')\left[1 - \frac{ik}{w}\bar{\rho}_\ell(z')\right] \cos w(1-z')\, dz' \\ & + \frac{k}{w}\int_0^1 \bar{\rho}_\ell(z') \sin wz' \cos w(1-z')\, dz' \\ & - \frac{i\gamma}{w}\int_0^1 \bar{V}_\ell(z') \frac{1}{\varphi_0} \frac{dg^{(0)}}{\varphi_0}(z') \cos w(1-z')\, dz' \\ & - (3-\gamma)\int_0^1 \bar{V}(z') \frac{i\gamma g^{(0)}}{\varphi_0}(z') \sin w(1-z')\, dz' \\ & - (3-\gamma)\int_0^1 \bar{V}(z') \sin wz' \sin w(1-z')\, dz' \\ & - \gamma M(w)\int_0^1 e^{-iw\bar{\tau}_t(z')} \frac{d\bar{V}}{dz'}(z') \sin w(1-z')\, dz' \Big\} \end{aligned} \qquad \text{C 16}$$

The first, fourth and seventh terms may be expanded and then combined to yield:

$$\mathfrak{F}_A = w\cos w \int_0^1 \bar{V}(z')\,dz' - (2-\gamma)w\int_0^1 \bar{V}(z')\cos w(1-2z')\,dz' + \frac{k}{2}\int_0^1 \bar{\rho}_\ell(z')\left[\sin w - \sin w(1-2z')\right]dz' \qquad \text{C 17}$$

And, upon introducing the notation:

$$A_1(w) = \int_0^1 \bar{\rho}_\ell(z')\sin w(1-2z')\,dz'$$

$$A_2(w) = \int_0^1 \cos w(1-z')\,dz' \int_0^{z'} e^{-iw\bar{\tau}_t(z'')}\frac{d\bar{V}}{dz'}(z')\,dz'$$

$$A_3(w) = \int_0^1 \bar{V}(z')\cos w(1-2z')\,dz'$$

$$A_4(w) = \int_0^1 e^{-iw\bar{\tau}_t(z')}\frac{d\bar{V}}{dz'}(z')\sin w(1-z')\,dz' \qquad \text{C 18}$$

$$A_5(w) = \int_0^1 A(z')e^{-iw\bar{\tau}_t(z')}\cos w(1-z')\,dz'$$

$$A_6(w) = \int_0^1 A(z')e^{-iw\bar{\tau}_t(z')}\bar{\rho}_\ell(z')\cos w(1-z')\,dz'$$

$$A_7(w) = \int_0^1 A(z')e^{-iw\bar{\tau}_t(z')}\bar{V}(z')\sin w(1-z')\,dz'$$

$$A_8(w) = \int_0^1 \frac{dA(z')}{dz'}e^{-iw\bar{\tau}_t(z')}\bar{V}_\ell(z')\cos w(1-z')\,dz'$$

Eq. C 17 becomes:

$$\mathfrak{F}_A = w\cos w\int_0^1 \bar{V}(z')\,dz' - (2-\gamma)wA_3(w) + \frac{k}{2}\sin w\int_0^1 \bar{\rho}_\ell(z')\,dz' - \frac{k}{2}A_1(w) \qquad \text{C 19}$$

while Eq. C 18 may be written:

$$A(w) = \mathfrak{F}_A - \gamma w A_2(w)H(w) + \gamma w^2 A_2(w)G(w)\,im\bar{\delta} + J(w)\left[-2wA_5(w) + ikA_6(w) - (3-\gamma)iwA_7(w) - iA_8(w)\right] - \gamma w M(w)A_4(w) \qquad \text{C 20}$$

At this point, we may insert Eqs. C 2 , C 14 and 10.32 into Eq. 12.14 and

we obtain:

$$h_1 m\bar{\delta} = h_2 n(1 - e^{-i\omega\bar{\delta}}) + h_3 \qquad 13.1$$

where

$$h_1 = \gamma G(\omega)\left[\omega^2 B_2(\omega) - \omega L(\omega)\right] + \gamma \bar{V}_e \alpha_n \omega^2 A_2(\omega) G(\omega) \qquad C\ 21$$

$$h_2 = -i\gamma \bar{V}_e \alpha_n C(\omega) - \gamma D(\omega) \qquad C\ 22$$

and

$$\begin{aligned} h_3 = {} & \bar{V}_e \alpha_n (\cos\omega + ik\sin\omega) \\ & + \bar{V}_e \alpha_n i \Big\{ \beta_A - \gamma\omega A_2(\omega) H(\omega) + J(\omega)\big[-2\omega A_5(\omega) \\ & + k i A_6(\omega) - (3-\gamma) i\omega A_7(\omega) - i A_8(\omega)\big] - \gamma\omega M(\omega) A_4(\omega) \Big\} \\ & + \bar{V}_e \beta_n \left(\frac{\gamma\epsilon}{\varphi_0}\right)^{(0)} + i\sin\omega + \gamma \bar{V}_e \cos\omega \\ & - \gamma H(\omega) L(\omega) - \beta_B + \gamma\omega B_2(\omega) H(\omega) \\ & - J(\omega)\big[-2\omega B_5(\omega) + ik B_6(\omega) + (3-\gamma) i\omega B_7(\omega) \\ & \quad - i B_8(\omega)\big] - \gamma\omega M(\omega) B_4(\omega) . \end{aligned} \qquad C\ 23$$